Lecture 2
(Simple Harmonic Motion)

Physics 262-01 Spring 2019

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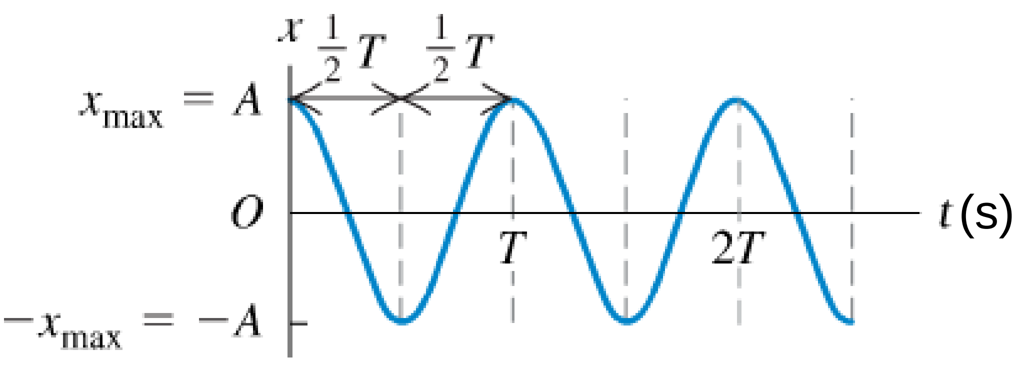
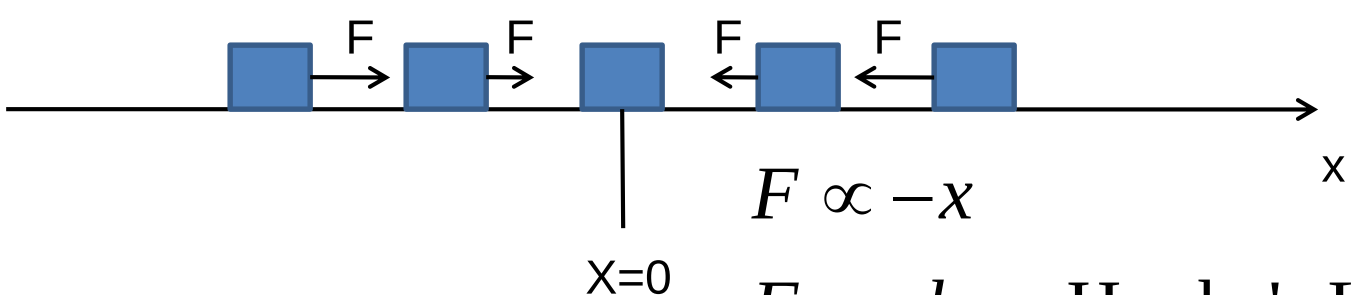
Reading Quiz

- Can the equation $x(t) = x_{\max}\cos(t)$ describe the graph below?
    A) Yes
    B) No
    C) Depends on what T is.



Simple Harmonic Motion

- A type of periodic motion with a very explicit definition:
- Motion about an equilibrium point with a restoring force proportional to the distance away from the equilibrium point.

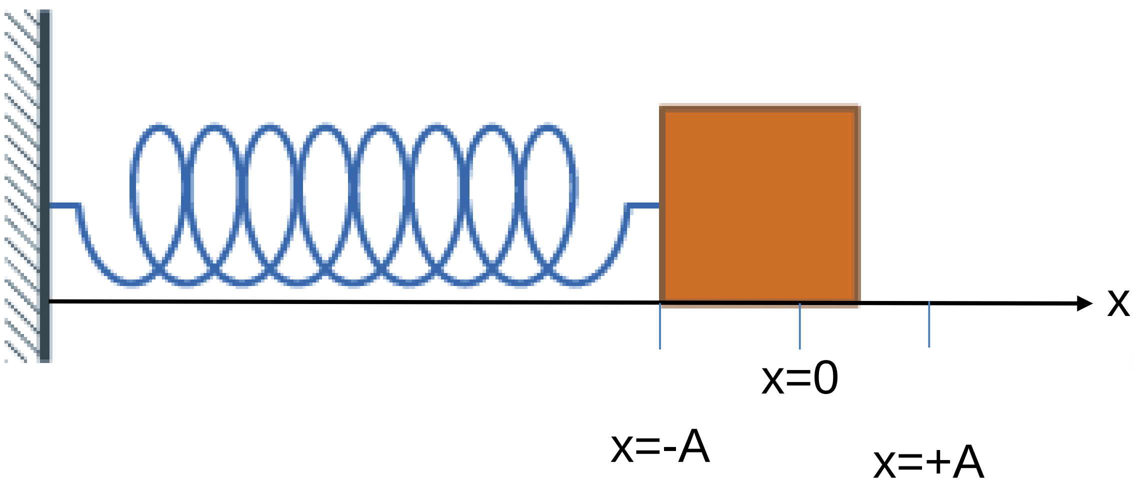


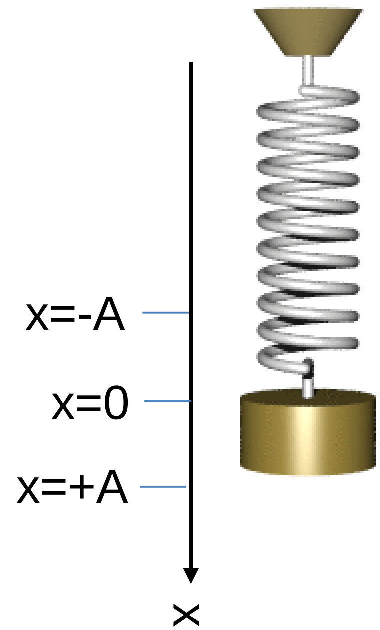
$$F \propto -x$$

$$F = -kx \quad \text{Hooke's Law}$$

Simple Harmonic Motion

$$F_{Net} = -kx$$





Simple Harmonic Motion

- Analyze: $F = -kx \Rightarrow$ Hooke's Law

$$ma = -kx \Rightarrow \quad \text{Newton's 2nd Law}$$

$$m\frac{d^2x}{dt^2} = -kx \Rightarrow \quad \text{Definition of acceleration}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \text{Divide both sides by m}$$

- Differential equation relating the changing acceleration to the position.
- Try non-periodic solutions:

$$x(t) = C \Rightarrow \frac{d^2x}{dt^2} = 0 \neq -\frac{k}{m}x(t) \quad \text{unless } C = 0$$

$$x(t) = e^{\sqrt{\frac{k}{m}}t} \Rightarrow \frac{dx}{dt} = \sqrt{\frac{k}{m}}e^{\sqrt{\frac{k}{m}}t} \Rightarrow \frac{d^2x}{dt^2} = \frac{k}{m}e^{\sqrt{\frac{k}{m}}t} \neq -\frac{k}{m}e^{\sqrt{\frac{k}{m}}t}$$

Simple Harmonic Motion

- Try a periodic solution: $$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$x(t) = \cos(ct) \Rightarrow \frac{dx}{dt} = -c\sin(ct) \Rightarrow \frac{d^2x}{dt^2} = -c^2\cos(ct) = -\frac{k}{m}\cos(ct)$$

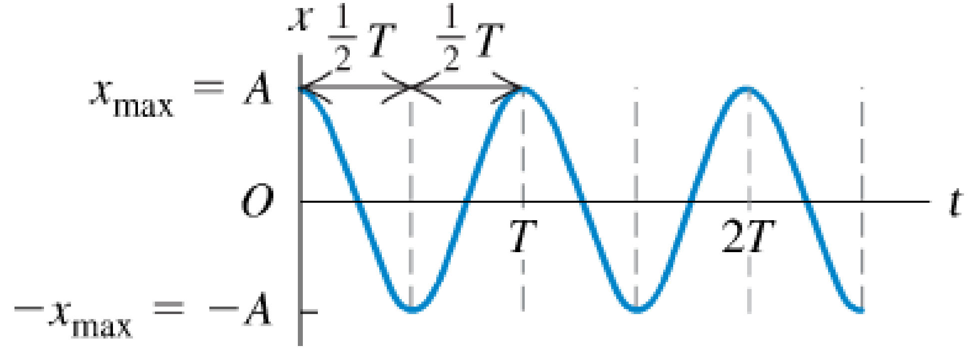
$$\text{if } c^2 = \frac{k}{m}$$

- The general solution is:

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t + \phi\right)$$

or, equivalently,

$$x(t) = B\cos\left(\sqrt{\frac{k}{m}}\,t\right) + C\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

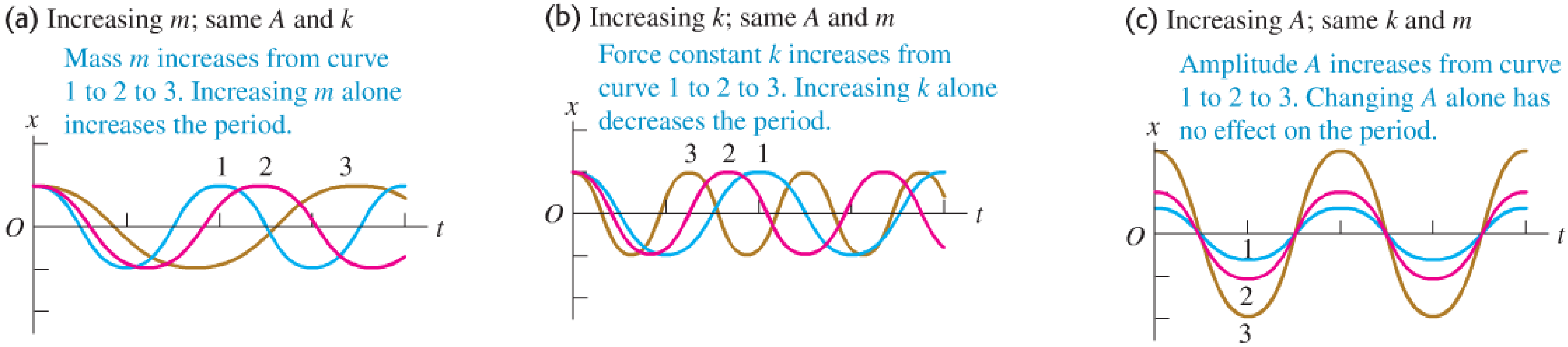


For φ = 0

Simple Harmonic Motion

- The factor in front of time sets the frequency of oscillations, so:

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t+\phi\right) = A\cos(\omega t+\phi), \quad \omega = \sqrt{\frac{k}{m}}$$



For φ = 0

Frequency, Angular Frequency and Period

- There is sometimes confusion about these quantities.

$$x(t) = A\cos(\omega t + \phi), \quad \omega = \sqrt{\frac{k}{m}}$$

- ω is called the angular frequency.
- The function x(t) returns to its starting point when ωt = 2π, so the period (amount of time to complete one cycle), is:

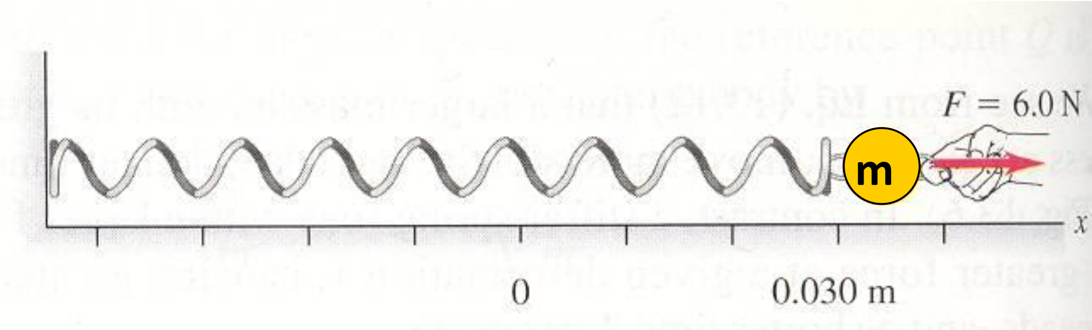
$$T = \frac{2\pi}{\omega}$$

- The frequency (number of cycles per second) is just:

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \Rightarrow \quad \omega = 2\pi f$$

Clicker Quiz

- **A mass m = 1.0kg is attached to a massless spring. The spring is stretched with a force of 6.0N to a distance of 0.03m and then released. What is the frequency of oscillations of the mass?**



A) 1.25 Hz.

B) 2.25 Hz.

C) 3.25 Hz.

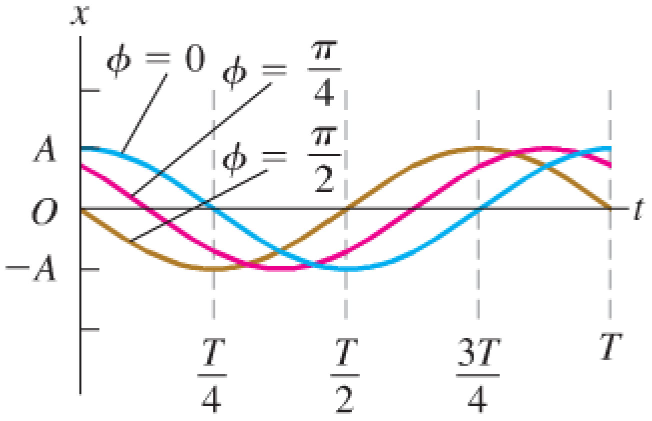
D) Not enough information to solve.

Simple Harmonic Motion

- The phase factor determines the value of x at t=0:

$$x(t) = A\cos(\omega t + \phi), \quad \omega = \sqrt{\frac{k}{m}}$$

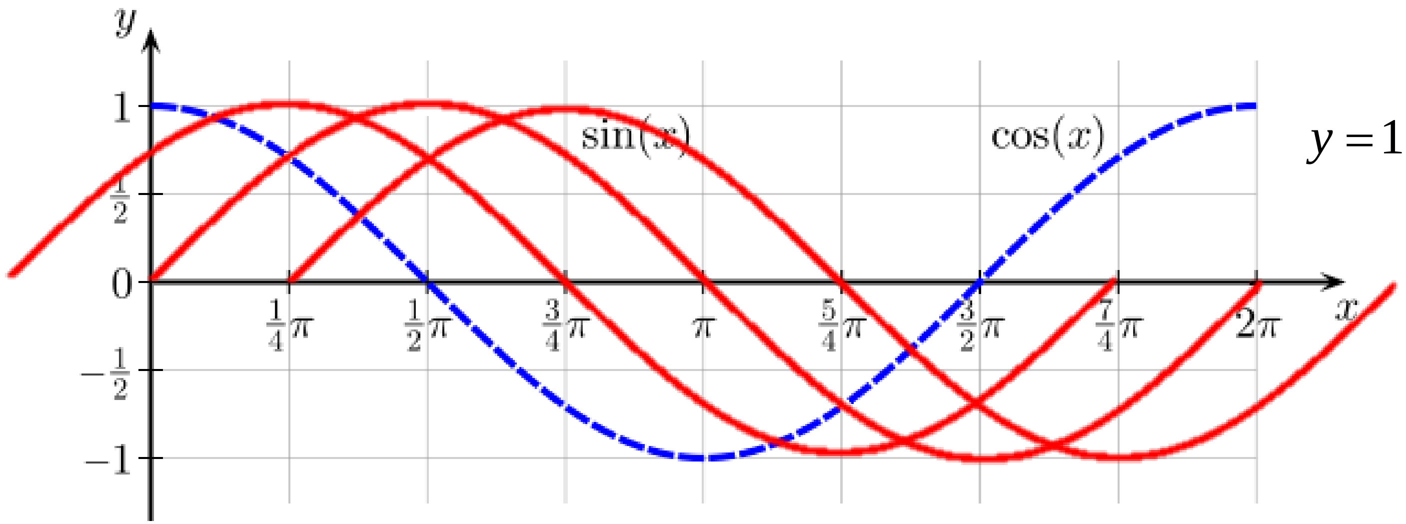
These three curves show SHM with the same period T and amplitude A but with different phase angles ϕ.



$$\phi = \text{phase angle that cycle is moved to left}$$
$$= 2\pi \times \text{ fraction of period moved to left}$$

Phase Angle

What if we want a sine function, but would like it to start at ¼ π instead of zero?



$$y = 1 \cdot \sin\left(x - \frac{1}{4}\pi\right)$$

What if we want a sine function, but would like it to start at -¼ π instead of zero?

$$y = 1 \cdot \sin\left(x + \frac{1}{4}\pi\right)$$

Position, Velocity and Acceleration

- We can differentiate to get the velocity

$$x(t) = A\cos(\omega t + \phi) \Rightarrow$$

$$v(t) = \frac{dx(t)}{dt} = -\omega A\sin(\omega t + \phi)$$

- And again to get acceleration

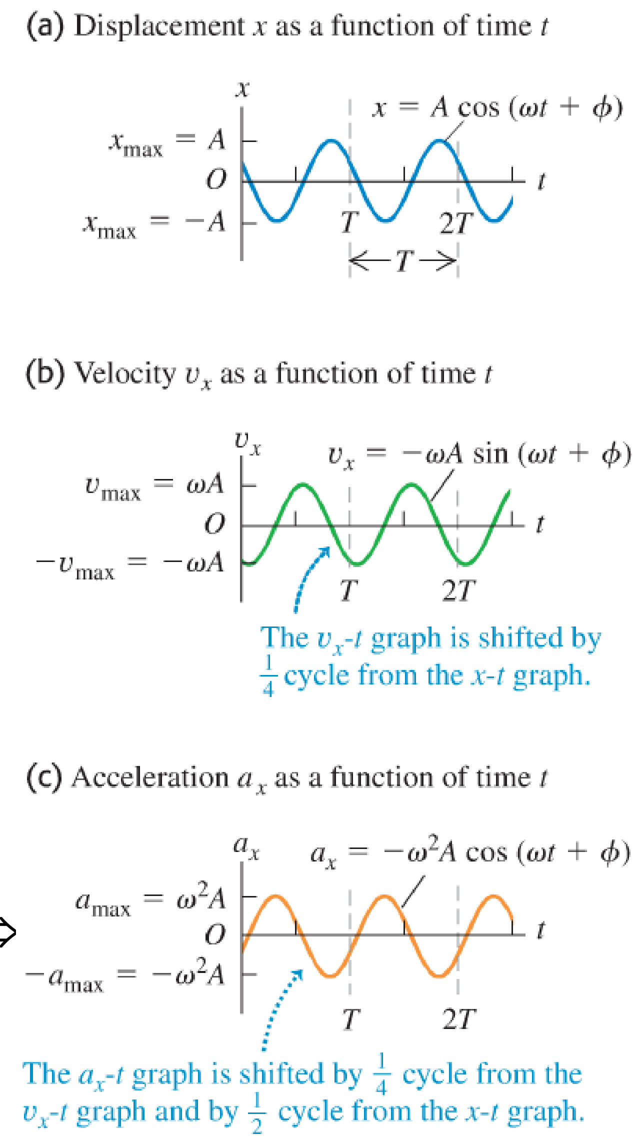
$$v(t) = -\omega A\sin(\omega t + \phi) \Rightarrow$$

$$a(t) = \frac{dv(t)}{dt} = -\omega^2 A\cos(\omega t + \phi)$$

- Note that:

$$a(t) = -\omega^2 A\cos(\omega t + \phi) = -\omega^2 x(t) = -\frac{k}{m}x(t) \Rightarrow$$

$$ma(t) = -kx(t)$$



Energy in Simple Harmonic Motion

- Without any other forces (friction), we can describe the energy of a spring-mass system by the kinetic energy:

$$KE = \frac{1}{2}mv^2(t) = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2}m\left(\sqrt{\frac{k}{m}}\right)^2 A^2 \sin^2(\omega t + \phi) = \frac{1}{2}kA^2 \sin^2(\omega t + \phi)$$

- And the potential energy is:

$$U_{el} = \frac{1}{2}kx^2(t) = \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$$

Energy in Simple Harmonic Motion

- So, the total energy is the sum of these:

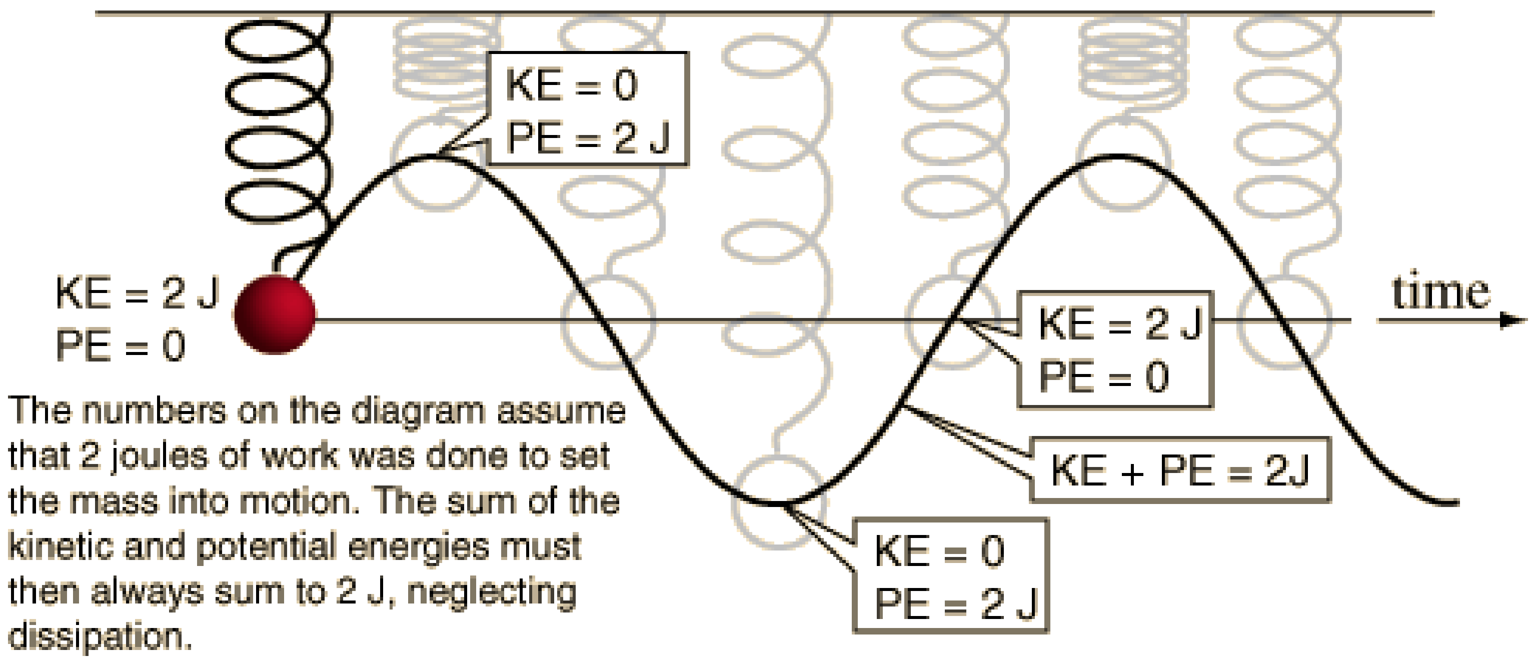
$$E_{total} = KE + U_{el} = \frac{1}{2}kA^2 \sin^2(\omega t + \phi) + \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$$

$$= \frac{1}{2}kA^2 \left(\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right)$$

$$= \frac{1}{2}kA^2$$

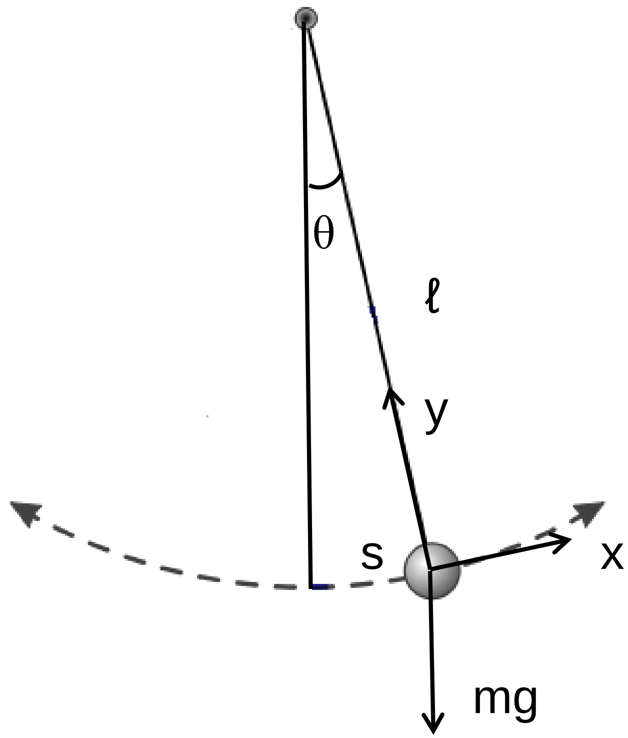
- But there is no time dependence here – conservation of energy!

Energy in Simple Harmonic Motion



No gravity…

The Simple Pendulum



$$F_x = -mg\sin\theta$$

For small θ:

$$F_x = -mg\sin\theta = -mg\theta = -mg\frac{s}{\ell}$$

$$= -\frac{mg}{\ell}s = -ks, \quad k = \frac{mg}{\ell}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{\frac{mg}{\ell}}{m}} = \sqrt{\frac{g}{\ell}}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}}$$

Torsion Pendulum

- Torsion spring applies a torque that is proportional to the angular displacement:

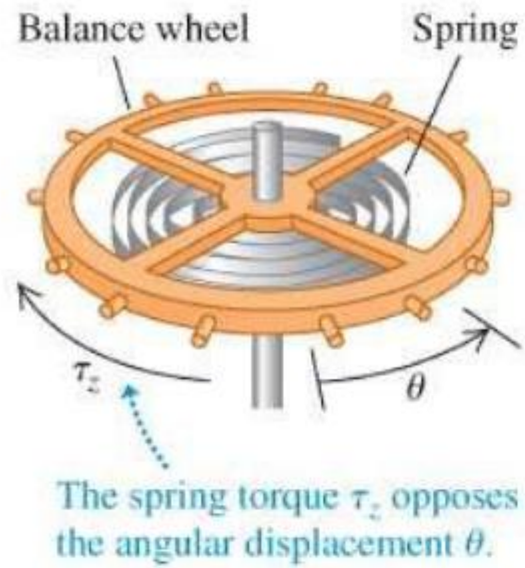
$$\tau \propto -\theta \Rightarrow \tau = -\kappa\theta$$

- From the rotational version of Newton's second law:

$$\sum \tau = I\alpha = I\frac{d^2\theta}{dt^2} \Rightarrow$$

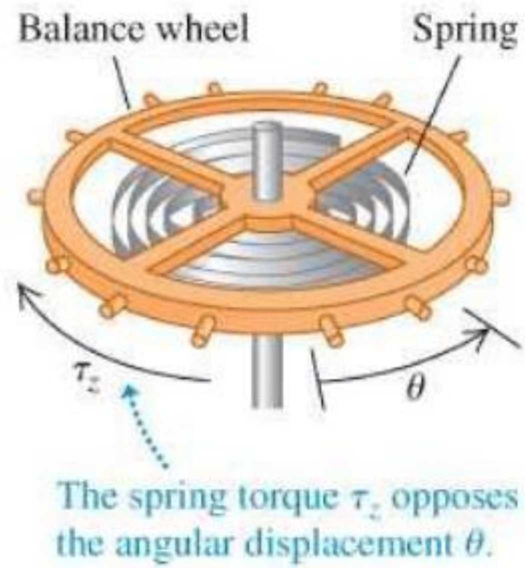
$$-\kappa\theta = I\frac{d^2\theta}{dt^2} \Rightarrow$$

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta$$



Torsion Pendulum

- But this is the same differential equation we had for a linear mass-spring system!



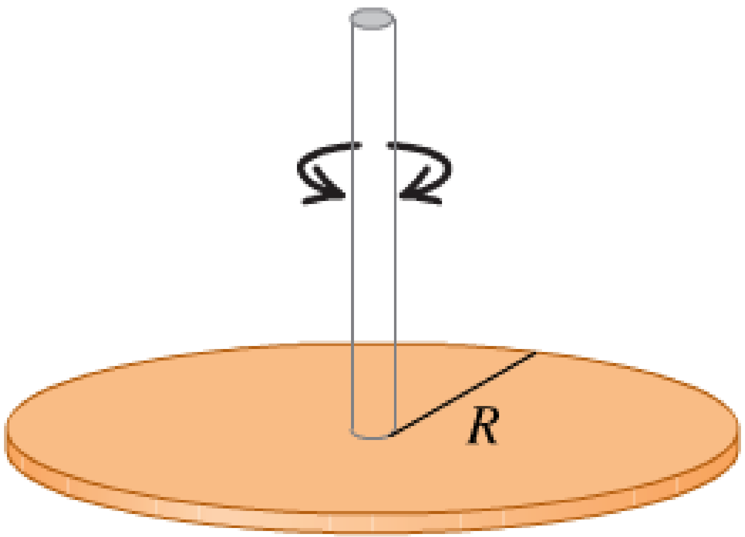
$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta \longleftrightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

- So, it has the same solutions:

$$\theta(t) = \Theta\cos(\omega t + \phi), \quad \omega = \sqrt{\frac{\kappa}{I}}$$

Problem 13.36

13.36. A thin metal disk with mass 2.00×10^{-3} kg and radius 2.20 cm is attached at its center to a long fiber (Fig. 13.32). The disk, when twisted and released, oscillates with a period of 1.00 s. Find the torsion constant of the fiber.



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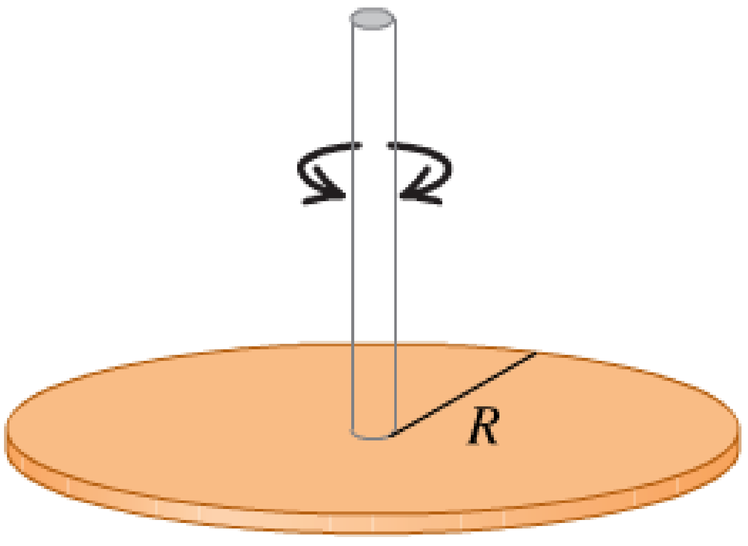

$$\lambda = \frac{\text{mass}}{\text{unit area}} = \frac{M}{A} = \frac{M}{\pi R^2}$$

$$dI_{\text{ring}} = dm \cdot r^2 = \left[\lambda\left(2\pi r\right)dr\right]r^2 \rightarrow$$

$$I_{\text{disk}} = \int_0^R dI_{\text{ring}} = \int_0^R \left[\lambda\left(2\pi r\right)dr\right]r^2 = \lambda 2\pi \int_0^R r^3 dr$$

$$I_{\text{disk}} = \lambda 2\pi \frac{R^4}{4} = \frac{M}{\pi R^2} \times 2\pi \frac{R^4}{4} = \frac{1}{2}MR^2$$

$$\omega = 2\pi f = \frac{2\pi}{T} = \sqrt{\frac{\kappa}{I}} \Rightarrow$$

$$\kappa = I\left(\frac{2\pi}{T}\right)^2 = \frac{1}{2}MR^2\left(\frac{2\pi}{T}\right)^2$$

$$\kappa = \frac{1}{2}\left(2\times10^{-3}kg\right)\left(0.022m\right)^2\left(\frac{2\pi}{1s}\right)^2$$

$$\kappa = 1.91\times10^{-3}\frac{kg\,m^2}{s^2} = 1.91\times10^{-3}\,Nm$$