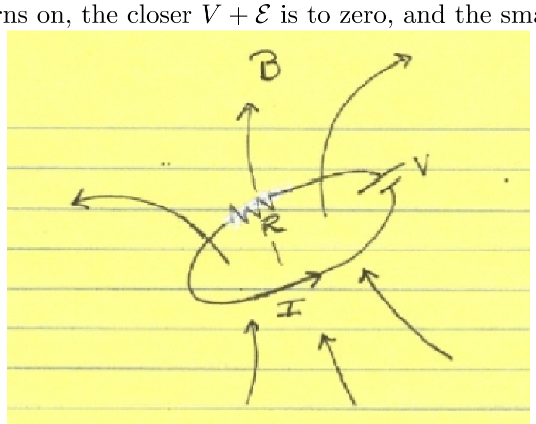
Physics 1320: Homework #13
Due Friday, 04/29/2022

Inductance

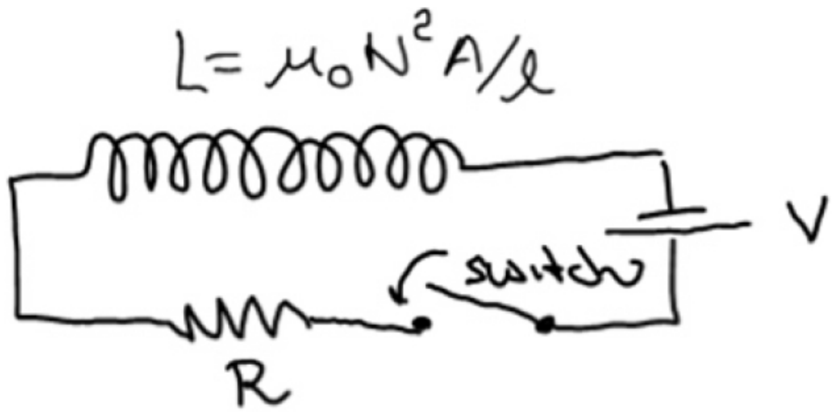
Please read chapter 30 in the following order. First, section 30.2. Then read 30.1, and then 30.3. Finally, read 30.4-30.6. Pass in solutions to the 7 problems below.

1 No Instantaneous Change in Current

Faraday's law comes into play whenever you flip a switch to close a circuit. Suppose you are connecting a battery with a voltage V to a resistor with resistance R, in a closed loop. Initially there is no current in the loop, but after the switch is closed, the current is given by $I = V/R$. Can the current change abruptly from 0 to V/R? This makes no sense, and here is the reason. The current will produce a magnetic field that pierces through the loop, giving a magnetic flux Φ. If the current were to change abruptly, from 0 to I, then the flux would also change abruptly, from 0 to Φ. In such a case, since $\mathcal{E} = -d\Phi/dt$, there would be an enormous pulse of emf, coming from the abrupt increase magnetic flux when the switch is thrown. According to Lenz's Law, the emf pulse would oppose the battery. This is why it is called "back-emf". According to Ohm's law, we should then have a reduction in the current, so that $I = (V + \mathcal{E})/R$, keeping in mind that $\mathcal{E}$ will be negative as you turn on the circuit. The faster the current turns on, the closer $V + \mathcal{E}$ is to zero, and the smaller will be the current.



To make this quantitative, let us consider a circuit for which it is easy to calculate the back-emf. Suppose that we place a long solenoid in series with the resistor and the battery, as shown in the figure.



According to Kirchoff's loop rule, the sum of the voltages around a closed loop is zero. As we have discussed above, this rule is no longer true in a situation where the magnetic flux is changing. Instead of writing

$$V - IR = 0$$

we must now write

$$V - IR = \frac{d}{dt}\Phi \tag{1}$$

Let us call this the "modified Kirchoff voltage law". The flux $\Phi = NBA$ where N is the number of turns in the solenoid and A is the cross sectional area of the solenoid. For a solenoid of length ℓ, the field

$$B = \mu_0 NI/\ell \tag{2}$$

is proportional to the number of windings per unit length N/ℓ, and the current I in each winding. Inserting (2) into (1), we obtain an equation describing the rate of change of current with time,

$$V - IR = \left(\mu_0 N^2 A/\ell\right)\frac{dI}{dt}. \tag{3}$$

The solenoid is also known as an inductor, and the combination of constants $\mu_0 N^2 A/\ell$ is called the inductance. While consideration of the solenoid made it easy to calculate the back-emf, the effect would have been present even without the solenoid, since any closed circuit consists of a wire loop that sustains flux. The *self*-inductance for any closed loop circuit is defined as the ratio of the magnetic flux to the current,

$$L = \frac{\Phi}{I}.$$

Its units are T·m^2/A. In SI, this is called a **Henry**;

$$1\ \mathrm{H} = 1\ \mathrm{T}\cdot\mathrm{m}^2/\mathrm{A}$$

We have seen that for a solenoid with N turns and length ℓ, it is straightforward to show that

$$L = \mu_0 N^2 A/\ell.$$

While the expression for L will vary for other geometries, it will always be given by N^2 multiplying some function of the dimensions.

Introducing the inductance generalizes our discussion of Kirchoff's loop rule. As we have discussed above, for a closed loop circuit, the sum of the voltages is not zero, but is given by the modified Kirchoff law,

$$V - IR = \frac{d}{dt}\Phi$$

Using the expression $\Phi = LI$, we can factor out L, and write

$$V - IR = L\frac{dI}{dt} \tag{4}$$

There is always some inductance associated with any circuit. When you are only interested in the steady-state, you set $dI/dt = 0$, and you recover the usual Kirchoff voltage rule. But if you are interested in the turn-on or turn-off behavior, or if you are interested in ac currents, you keep it in. For the case of closing a switch, it is a matter of integrating equation (8) to find the current as a function of time. The result is,

$$I = \frac{V}{R}\left[1 - \exp\left(-t/\tau\right)\right]$$

where the time constant,

$$\tau = L/R,$$

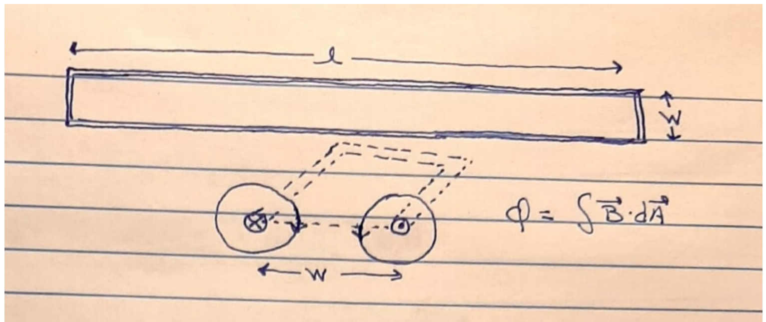
for the so-called "RL circuit", is the ratio of the inductance L to the resistance R.

Problem 1: Consider a narrow rectangular wire loop with length ℓ and width w carrying a current I. The wire comprising the loop has a cylindrical cross section, with radius a. By narrow, we mean that $\ell >> w >> a$.

(a) Using the fact that $B = \mu_0 I/2\pi r$ outside of a long straight wire, integrate to find the flux $\int \vec{B}\cdot d\vec{A}$ in the rectangular loop. Exclude the small region inside the wires, and neglect contributions of the flux coming from the far ends of the rectangle. Show that

$$\Phi \simeq \frac{\mu_0 \ell I}{\pi}\ln\left(\frac{w}{a}\right)$$

(b) Consider the narrow rectangular loop of wire made by high-voltage transmission lines. Take $\ell = 300$ km, the distance between Albuquerque and the coal-fired electrical generator in Farmington. Let $w = 5$ m, the distance between wires for a typical high-voltage power transmission line. Take the radius of the wires $a \simeq 1$ cm. What is the self-inductance L?



Problem 2: A closed loop circuit is comprised of an 8.5 H inductor in series with a 600 Ω resistor, together in series with a 10 volt battery. A flip of a switch closes the circuit. Obtain an expression giving the current I in milliamps as a function of time t in milliseconds (see textbook page 1000, equation 30.14) and use the computer to make a graph showin I versus t in the region where this behavior is interesting - from 0 to 50 ms. What is the time constant for the circuit? What is I after 30 ms? What is I when steady-state is reached?

2 Mutual Inductance and Transformers

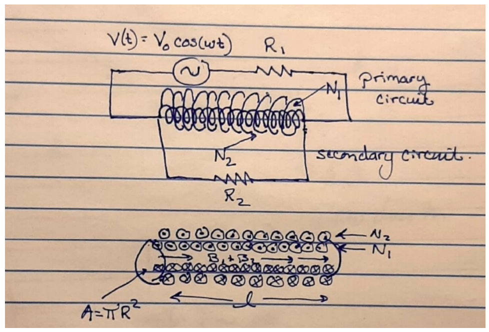
It is possible to couple two closed-loop circuits together by their magnetic flux alone. The next problem provides an illustration. Your textbook opens chapter 30 with a discussion of mutual inductance. You may find it helpful to read this first.

Problem 3: Consider two circuits coupled together by a transformer. The primary circuit is driven by an ac voltage source $V(t) = V_0 \cos \omega t$ in series with a resistor R_1 and an inductor consisting of a long solenoid with length ℓ, cross sectional area A, and N_1 turns of insulated wire wrapped around a core having linear permeability μ. The secondary circuit consists of a resistor R_2 in a closed loop. The secondary circuit is in "magnetic contact" with the primary circuit, in the sense that the wire in the secondary circuit is wrapped N_2 times around the same solenoidal core. The total magnetic field inside the solenoid is the sum $B_1 + B_2$, where B_1 is the field produced by the current I_1 in the primary, and B_2 is the field produced by the current I_2 in the secondary.

(a) Write down an expression for B_1 in terms of I_1. Write down an expression for B_2 in terms of I_2.

(b) The extended Kirchoff voltage law for the primary circuit is $V(t) - I_1 R_1 = \frac{d}{dt}\Phi_1$ where $\Phi_1 = (B_1 + B_2)N_1 A$. The extended Kirchoff voltage law for the secondary circuit is $-I_2 R_2 = \frac{d}{dt}\Phi_2$ where $\Phi_2 = (B_1 + B_2)N_2 A$. Combine these equations and eliminate $\frac{d}{dt}(B_1 + B_2)$, so as to show that $V(t) - I_1 R_1 = -\frac{N_1}{N_2} I_2 R_2$.

(c) Suppose that the voltage dropped across your house is $I_2 R_2$, and that $V(t) - I_1 R_1$ is the voltage in the transmission lines in your neighborhood. What should the turns ratio N_2/N_1 be so that 13,000 volts RMS in the transmission line is converted down to 240 volts RMS?



3 Energy is Stored in an Inductor

Suppose that we connect an inductor to a battery in a closed loop circuit. Suppose also that the wires are superconducting, having no resistance. In such a case, the voltage V across the battery will only be opposed by the back-emf, LdI/dt, across the inductor, so that

$$V = L\frac{dI}{dt}.$$

Let us now multiply both sides of this equation by I :

$$VI = LI\frac{dI}{dt}$$

On the left we recognize the power $P = VI$ supplied by the battery. It follows that the term on the right is the power absorbed by the inductor. Let us multiply both sides of the equation by dt, and integrate over time to find the total energy stored in the inductor after the circuit has been left on for some time.

$$\begin{aligned} \int P dt &= \int LI\frac{dI}{dt}dt \\ &= L\int IdI = \frac{1}{2}LI^2 \end{aligned}$$

Since the inductance L is a constant that depends only on geometry, it can be brought outside the integral. The dt cancels, leaving us with a perfect differential in the current, IdI, which integrates to $I^2/2$. Thus we obtain the famous result that the energy stored in an inductor having a current I is given by

$$U = \frac{1}{2}LI^2$$

This reminds us of the equally famous expression for the energy stored in a capacitor,

$$U = \frac{1}{2}\frac{Q^2}{C}$$

It appears that current is to an inductor as charge is to a capacitor - in terms of energy at least.

For a solenoid, it is easy to rewrite the expression for energy in terms of the magnetic field. Since $B = \mu NI/\ell$, and $L = \mu N^2 A/\ell$, we can write

$$\begin{aligned} U &= \frac{1}{2}\left(\mu N^2 A/\ell\right)\left(\frac{B\ell}{\mu N}\right)^2 \\ &= \frac{1}{2\mu}B^2 A\ell \end{aligned}$$

Notice that the number of windings drop out, and that the product $A\ell$ is the volume of the solenoid. This is an illustration of a more general result. It can be shown quite generally that the total energy required to build up the currents to create a magnetic field of any sort is given by a volume integral,

$$U = \frac{1}{2}\int_{\text{all space}} \mu^{-1} B^2 d^3 r$$

of the square of the magnetic field. The integral is to be taken over all space; in applications, of course, it only needs to cover the volume in space where B is not zero.

As we have seen earlier this semester, we can write the energy stored in a parallel plate capacitor in an analogous fashion. Since $Q = CV = CEd$, and $C = \epsilon A/d$, it follows that

$$\begin{aligned} \frac{1}{2}\frac{Q^2}{C} &= \frac{1}{2}\frac{(CEd)^2}{C} \\ &= \frac{1}{2}\frac{(\epsilon A/d)^2 E^2 d^2}{(\epsilon A/d)} \\ &= \frac{1}{2}\epsilon E^2 Ad \end{aligned}$$

Notice that the product Ad is the volume of the capacitor. This is also an illustration of a more general result. The energy required to build up an electric field of any sort is given by a volume integral,

$$U = \frac{1}{2}\int_{\text{all space}} \epsilon E^2 \; d^3 r,$$

taken over all space. For combined electric and magnetic fields, the total energy is given by the following integral,

$$U = \frac{1}{2}\int_{\text{all space}} \left(\epsilon E^2 + \mu^{-1} B^2\right) \; d^3 r.$$

The integrand has units of energy per unit volume, and is called the *electromagnetic energy density*,

$$u = \frac{1}{2}\left(\epsilon E^2 + \mu^{-1} B^2\right)$$

The phrase "energy density" conjures up the notion that there is something real, occupying space, wherever the fields E and/or B are nonzero.

Problem 4: How much energy is stored when a 100 μF capacitor has a charge of 50 mC? If this energy is transferred to an inductor with inductance 1.0 H, what is the current in the inductor?

Problem 5: A capacitor C is connected to an inductor L in a closed loop circuit. Initially the capacitor has a charge Q_0, and the current is zero. The modified Kirchoff loop rule for this circuit is

$$-\frac{Q}{C} = L\frac{dI}{dt}.$$

(a) Rewrite this equation in terms of charge only, by expressing the current in terms of the rate of change of the charge on the capacitor. Show that the charge obeys the following equation:

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0.$$

(b) This is *isomorphic* to the equation for a mass m connected to a spring with stiffness constant k, moving along the x axis, i.e.,

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0,$$

except that charge Q plays the role of the displacement x. Recall that a mass connected to a spring oscillates, with a natural frequency $\omega_0 = \sqrt{k/m}$. By comparison, if $L = 1$ mH, and $C = 10$ μF, what is the natural frequency of oscillation of this LC series circuit?

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i·so·mor·phic

/ˌīsəˈmôrfik/

adjective

corresponding or similar in form and relations.

- having the same crystalline form.

(c) The solution to the equation in (a) is $Q(t) = Q_0 \cos(\omega_0 t)$. If $L = 1$ mH, and $C = 10$ μF, and $Q_0 = 1.0$ mC, what is the maximum voltage across the inductor? What is the maximum voltage across the capacitor? How is the voltage across the capacitor related to the voltage across the inductor?

Problem 6: The magnetic field strength in a long superconducting nioboium-titanium wound solenoid buried underground in Switzerland is 7.7 T. The cross sectional area of the solenoid is 3.0×10^{-4} m^2, and its length is approximately 27 km. The magnet is cooled by 96 tons of liquid helium to 1.9 K. If a loss of cooling allows the superconducting wires to suddenly become normal, so that B drops to zero, how much magnetic energy will be released as heat? Take $\mu = \mu_0$.

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