

# Physics 566 - Quantum Optics II

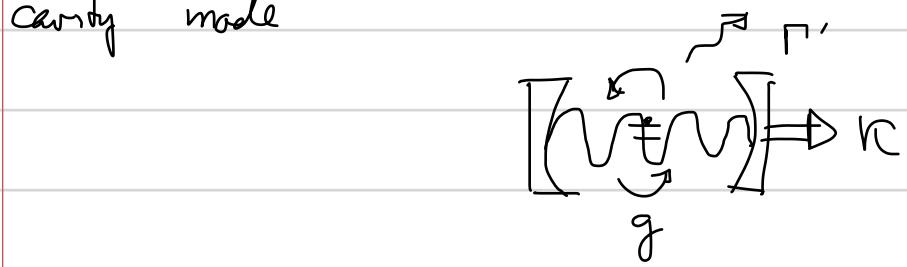
## Problem Set 9 - Solutions

### Problem 1: the Purcell Effect

The Purcell Effect is named after Ed Purcell, one of the great AMO physicists of the mid 20<sup>th</sup> century, who showed how the existence of boundary conditions, e.g. those associated with mirrors in an E+M cavity, could affect the spontaneous emission of atoms in a seminal paper, Phys. Rev. 69, 681 (1946).

It is not a surprise that boundary conditions effect radiating systems - we are familiar with this classically. The microscopic charges in the material that make up the boundary react to a neighboring dipole. The details of this microscopic response are summarized by the boundary conditions. The interesting part is that this applies at the quantum level. The modes of the quantum electromagnetic field must satisfy the boundary conditions. This affect spontaneous emission.

We seek the rate of emission of an atom into a given cavity mode. This only makes sense in the "weak coupling regime" in which the rate of decay of photons due to absorption in and/or transmission through the cavity mirrors is much faster than the rate of coherent Rabi oscillations between the atom and cavity mode.



$$\hbar g = \sqrt{\frac{2\pi\hbar\omega_c}{V}} \vec{U}_c(\vec{R}) \cdot \vec{J}_{\text{deg}}$$

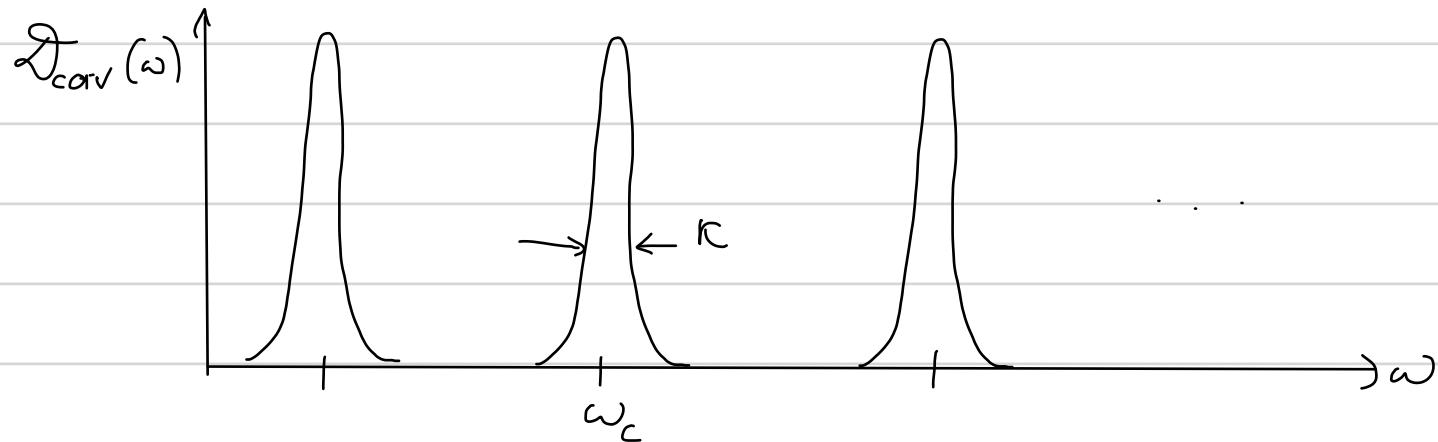
Vacuum Rabi freq,  
 $\kappa = \frac{1}{L_{\text{cavity}}} \gg g$

cavity mode @  
atom's position

$\Gamma'$  = spontaneous emission rate in to all other modes

(a) One way to calculate the Purcell effect is to use Fermi's Golden Rule. We seek the rate of spontaneous emission into a given cavity mode. This notion only makes sense in the weak coupling regime, otherwise photons emitted

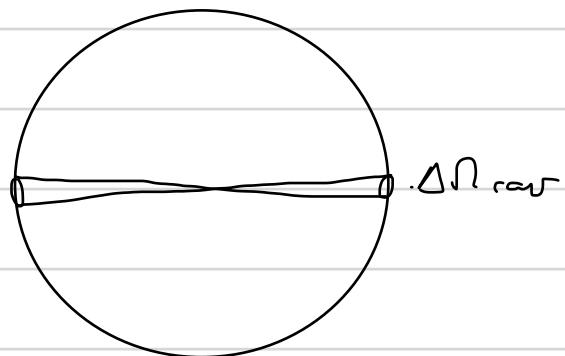
into the cavity will get reabsorbed in a coherent Rabi oscillation; spontaneous emission is irreversibly decay. The effect of cavity is to modify the density of states of electromagnetic modes available for the atom in which it can radiate. The density of states in directions captured by the cavity has resonance peaks



Near a particular cavity resonance  $\omega_c$ , the density of states is Lorentzian

$$D_{\text{cav}}(\omega) = \frac{\kappa/2\pi}{(\omega - \omega_c)^2 + \kappa^2/4}$$

Note, this is the density of states around the frequency band near  $\omega_c$ , and for modes propagating within the solid angle subtended by the cavity.



Emission into solid angles  $4\pi - \Delta\Omega_{\text{cav}}$  near the atomic resonance defines  $\Gamma'$ . Generally  $\Delta\Omega \ll 4\pi$ , i.e. the cavity modes are "paraxial." In this case,  $\Gamma' \approx \Gamma_{\text{free}}$ , the free-space spontaneous emission rate.

With this background, we can write the spontaneous emission rate into the cavity mode using Fermi's Golden Rule

$$\Gamma_{\text{cav}} = \frac{2\pi}{\hbar^2} |\langle \Psi_{\text{final}} | \hat{H}_{\text{int}} | \Psi_{\text{initial}} \rangle|^2 D_{\text{cav}}(\omega_{\text{eg}}) \quad (\text{Next Page})$$

Where  $|\Psi_{\text{initial}}\rangle = |e, 0\rangle$ ,  $|\Psi_{\text{final}}\rangle = |g, 1\rangle$  one photon in the cavity

$$\hat{H}_{\text{int}} = \hbar g (\hat{a}_c^\dagger \hat{a}_+ + \hat{a}_c \hat{a}_-)$$

$$\Rightarrow \Gamma_{\text{cav}} = \frac{2\pi}{\hbar V} (\hbar g)^2 \mathcal{D}_{\text{cav}}(\omega_{eg}) = \left( \frac{4\pi^2 \omega_{eg}}{\hbar V} \right) |\vec{u}_{\text{cav}}(\vec{R}) \cdot \vec{d}_{\text{eg}}|^2 \mathcal{D}_{\text{cav}}(\omega_{eg})$$

Now  $\Gamma_{\text{free}} = 2\pi \overline{|g(\omega_{eg})|^2} \mathcal{D}_{\text{free}}(\omega_{eg})$  is the free-space spontaneous emission rate

where  $\overline{|g(\omega_{eg})|^2} = \frac{16\pi^2 \omega_{eg}}{3\hbar V} |\vec{d}_{\text{eg}}|^2$  is the free space coupling constant, averaged

over all directions of propagation and polarizations,  $\mathcal{D}_{\text{free}}(\omega_c) = \frac{V}{(2\pi)^3 c^3} \frac{\omega_c^2}{\lambda_{eg}^3} = \left(\frac{V}{\lambda_{eg}^3}\right) \frac{1}{\omega_{eg}}$   
is the free-space density of states.

$$\begin{aligned} \text{Thus, } \frac{\Gamma_{\text{cav}}}{\Gamma_{\text{free}}} &= \frac{3}{8\pi} \frac{|\vec{u}_{\text{cav}}(\vec{R}) \cdot \vec{d}_{\text{eg}}|^2}{|\vec{d}_{\text{eg}}|^2} \frac{\mathcal{D}_{\text{cav}}(\omega_{eg})}{\mathcal{D}_{\text{free}}(\omega_{eg})} \\ &= \frac{3}{16\pi^2} \frac{\lambda_{eg}^3}{V} \frac{\omega_{eg}}{\kappa} \frac{\kappa^2}{(\omega_{eg} - \omega_c)^2 + \kappa^2/4} \frac{|\vec{u}_{\text{cav}}(\vec{R}) \cdot \vec{d}_{\text{eg}}|^2}{|\vec{d}_{\text{eg}}|^2} \end{aligned}$$

$$\Rightarrow \boxed{\Gamma_{\text{cav}} = \Gamma_{\text{free}} \mathcal{F} \frac{\kappa^2/4}{(\omega_{eg} - \omega_c)^2 + \kappa^2/4} \frac{|\vec{u}_{\text{cav}}(\vec{R}) \cdot \vec{d}_{\text{eg}}|^2}{|\vec{d}_{\text{eg}}|^2}}$$

where  $\mathcal{F} = \frac{3}{4\pi^2} \frac{\lambda_{eg}^3}{V} Q$   $Q$  = "Parcell factor",  $Q = \frac{\omega_{eg}}{\kappa} =$  Cavity "quality factor"

For  $Q \gg 1$  (e.g.  $10^3 - 10^6$ ) and  $V$  not too large compared to  $\lambda^3$ ,  $\mathcal{F} \gg 1$

(b) If the cavity is tuned to the atom resonance  $\omega_c = \omega_{eg}$ , and we the atom is positioned so that  $|\vec{u}_{\text{cav}} \cdot \vec{d}_{\text{eg}}| = |\vec{d}_{\text{eg}}| \Rightarrow \Gamma_{\text{cav}} = \mathcal{F} \Gamma_{\text{free}}$ ,  $\mathcal{F} \gg 1 \Rightarrow$  on resonance the decay rate into the cavity mode is enhanced. When

$$\Gamma_{\text{cav}} \approx \Gamma_{\text{free}} \mathcal{F} \frac{\kappa^2}{4\omega_c^2} = \Gamma_{\text{free}} \frac{3}{4\pi^2} \frac{\lambda^3}{V} Q \frac{1}{4Q^2} = \Gamma_{\text{free}} \frac{3}{16\pi^2 Q} \frac{\lambda^3}{V} : \text{emission into k_0 cavity is suppressed}$$

(c) We can recover this solution by explicitly solving the atom-photon dynamics in the Born-Markov (Wigler-Weisskopf) approximation.

We treat decay of the atom and cavity into other modes using a nonhermitian Hamiltonian

$$\hat{H}_{\text{eff}} = \hat{H} - i \frac{\hbar \Gamma'}{2} \hat{G}_+ \hat{G}_- - i \frac{\hbar \kappa}{2} \hat{a}_c^\dagger \hat{a}_c$$

In the interaction picture the atom-cavity coupling Hamiltonian is

$$\hat{H}_{\text{int}}^{(I)} = \hbar g (\hat{a}_c \hat{G}_+ e^{-i(\omega_c - \omega_{eg})t} + \hat{a}_c^\dagger \hat{G}_- e^{+i(\omega_c - \omega_{eg})t})$$

The interaction couples states only in 2D subspaces. We take  $|\Psi(0)\rangle = |g,0\rangle \Rightarrow$   
(in the interaction picture)  $|\Psi(+)\rangle = c_{g,1}(+) |g,1\rangle + c_{e,0}(+) |e,0\rangle$

$$\begin{aligned} \dot{c}_{g,1} &= -\frac{i}{\hbar} \langle g,1 | \hat{H}_{\text{eff}}^{(I)} | \Psi \rangle = -\frac{\kappa}{2} c_{g,1} - ig c_{e,0} e^{+i(\omega_c - \omega_{eg})t} \\ \dot{c}_{e,0} &= -\frac{i}{\hbar} \langle e,0 | \hat{H}_{\text{eff}}^{(I)} | \Psi \rangle = -\frac{\Gamma'}{2} c_{e,0} - ig c_{g,1} e^{-i(\omega_c - \omega_{eg})t} \end{aligned}$$

(d) As we did in free space, we can formally integrate

$$\Rightarrow c_{g,1}(+) = \cancel{c_{g,1}(0)} e^{-\frac{\kappa t}{2}} - ig \int_0^t dt' e^{-\frac{\kappa}{2}(t-t')} e^{i(\omega_c - \omega_{eg})t'} c_{e,0}(t')$$

$$\Rightarrow \dot{c}_{e,0} = -\frac{\Gamma'}{2} c_{e,0} - g^2 \int_0^t dt' e^{[-i(\omega_c - \omega_{eg}) - \frac{\kappa}{2}](t-t')} c_{e,0}(t') \Rightarrow c_{e,0}(+) \quad (\text{Born-Aprox.})$$

$$\Rightarrow \dot{c}_{e,0} \approx -\frac{\Gamma'}{2} c_{e,0} - g^2 \frac{1}{i(\omega_c - \omega_{eg}) + \frac{\kappa}{2}} c_{e,0}(0)$$

$$= \left( -\frac{\Gamma'}{2} - \pi g^2 \frac{\frac{\kappa}{2\pi}}{(\omega_c - \omega_{eg})^2 + \frac{\kappa^2}{4}} - ig^2 \frac{(\omega_c - \omega_{eg})}{(\omega_c - \omega_{eg})^2 + \frac{\kappa^2}{4}} \right) c_{e,0}$$

$$\frac{\Gamma_{\text{cav}}}{2} \qquad \qquad \qquad \frac{\delta E_{\text{cav}}}{\hbar}$$

Lamb shift  
change due to  
cavity.