

Phys 566, Quantum Optics

Problem Set #7 Solutions

Problem 1: Momentum and Angular Momentum in Field

$$\hat{P} = \int d^3x \left(\frac{\hat{\vec{E}}(\vec{x}) \times \hat{\vec{B}}(\vec{x})}{4\pi c} \right), \quad \hat{\vec{J}} = \int d^3x \left(\vec{x} \times \frac{\hat{\vec{E}}(\vec{x}) \times \hat{\vec{B}}(\vec{x})}{4\pi c} \right)$$

Quantized field

$$\hat{\vec{A}} = \sum_{\vec{k}, \lambda} \underbrace{\sqrt{\frac{2\pi\hbar\omega_k^2}{V\omega_k}}} \vec{e}_{k,\lambda} \hat{a}_{k,\lambda} e^{i\vec{k} \cdot \vec{x}} + h.c.$$

$$\hat{A}^{(+)} \qquad \qquad \qquad \hat{A}^{(-)}$$

$$\hat{\vec{E}} = +i \sum_{\vec{k}, \lambda} \underbrace{\sqrt{\frac{2\pi\hbar\omega_k}{V}}} \vec{e}_{k,\lambda} e^{i\vec{k} \cdot \vec{x}} \hat{a}_{k,\lambda} + h.c.$$

$$\vec{E}^{(+)} \qquad \qquad \qquad \vec{E}^{(-)}$$

$$\hat{\vec{B}} = +i \sum_{\vec{k}, \lambda} \underbrace{\sqrt{\frac{2\pi\hbar\omega_k}{V}}} \vec{k} \times \vec{e}_{k,\lambda} e^{i\vec{k} \cdot \vec{x}} \hat{a}_{k,\lambda} + h.c.$$

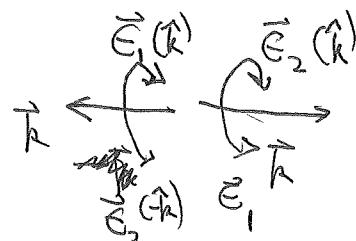
$$\vec{B}^{(+)} \qquad \qquad \qquad \vec{B}^{(-)}$$

Here I have used the basis of circularly polarized plane waves with orthogonality

$$\vec{e}_{k,\lambda}^* \cdot \vec{e}_{k',\lambda'} = \delta_{kk'} \delta_{\lambda\lambda'}$$

Note also

$$\vec{e}_{-\vec{k},\lambda_1} = \vec{e}_{\vec{k},\lambda_2}$$



(1a)

Let us plug the mode decomposition into \hat{P}

$$\Rightarrow \hat{P} = \int d^3x \left(\frac{\hat{E}^{(+)} \times \hat{B}^{(-)}}{4\pi c} + \frac{\hat{E}^{(+)} \times \hat{B}^{(+)} }{4\pi c} + h.c. \right)$$

Consider first term:

$$\begin{aligned} \int d^3x \frac{\hat{E}^{(+)} \times \hat{B}^{(-)}}{4\pi c} &= \sum_{\vec{k}, \lambda, \vec{k}', \lambda'} \frac{1}{4\pi c} (2\pi \hbar \sqrt{\omega_k \omega_{k'}}) \vec{e}_{\vec{k}, \lambda} \times (\hat{k}' \times \vec{e}_{\vec{k}', \lambda'}^*) \\ &\quad \underbrace{\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}') \cdot \vec{x}}}_{= S_{\vec{k}\vec{k}'}^{(3)}} \hat{a}_{\vec{k}, \lambda} \hat{a}_{\vec{k}', \lambda'}^+ \\ &\rightarrow = \sum_{\vec{k}} \sum_{\lambda, \lambda'} \frac{\hbar k}{2} \underbrace{[\vec{e}_{\vec{k}, \lambda} \times (\hat{k} \times \vec{e}_{\vec{k}, \lambda}^*)] \hat{a}_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda'}^+}_{\vec{k}(\vec{e}_{\vec{k}, \lambda} \cdot \vec{e}_{\vec{k}, \lambda}^*) - \vec{e}_{\vec{k}, \lambda}^* (\underbrace{\hat{k} \cdot \vec{e}_{\vec{k}, \lambda}}_{=0})} \\ &\quad \text{(transversality)} \end{aligned}$$

$$\Rightarrow \int d^3x \frac{\hat{E}^{(+)} \times \hat{B}^{(-)}}{4\pi c} = \frac{1}{2} \sum_{\vec{k}, \lambda} \hbar \vec{k} \hat{a}_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda}^+$$

Now, by similar steps

$$\begin{aligned} \int d^3x \frac{\hat{E}^{(+)} \times \hat{B}^{(+)} }{4\pi c} &= \sum_{\vec{k}} \sum_{\lambda, \lambda'} \frac{\hbar k}{2} \underbrace{(\vec{e}_{\vec{k}, \lambda} \times (-\hat{k} \times \vec{e}_{-\vec{k}, \lambda}^*))}_{-\hat{k}(\vec{e}_{\vec{k}, \lambda} \cdot \vec{e}_{-\vec{k}, \lambda}^*) + \vec{e}_{-\vec{k}, \lambda}^* (\underbrace{\hat{k} \cdot \vec{e}_{\vec{k}, \lambda}}_{=0})} \\ &= 0 \quad (\text{see page 4}) \end{aligned}$$

Thus,

$$\begin{aligned}\hat{P} &= \int d^3x \frac{\vec{E}^{(+)} \times \vec{B}^{(+)}}{4\pi c} + h.c. \\ &= \sum_{\vec{k}, \lambda} \hbar \vec{k} \left(\hat{a}_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda}^\dagger + \hat{a}_{\vec{k}, \lambda}^\dagger \hat{a}_{\vec{k}, \lambda} \right) \\ &= \sum_{\vec{k}, \lambda} \hbar \vec{k} \hat{a}_{\vec{k}, \lambda}^\dagger \hat{a}_{\vec{k}, \lambda} + \frac{\hbar}{2} \sum_{\lambda} \left(\sum_{\vec{k}} \vec{k} \right) \xrightarrow{\text{no mean momentum}} \\ \Rightarrow \boxed{\hat{P} = \sum_{\vec{k}, \lambda} \hbar \vec{k} \hat{a}_{\vec{k}, \lambda}^\dagger \hat{a}_{\vec{k}, \lambda}}\end{aligned}$$

Neat! in the plane wave decomposition, we see that the momentum in the field decompose into sum of photon momenta $\hbar \vec{k}$ times the number operator

$$\cancel{\hat{N}_{\vec{k}}} = \sum_{\lambda} \hat{a}_{\vec{k}, \lambda}^\dagger \hat{a}_{\vec{k}, \lambda} \quad \text{counting the number of photons with wave vector } \vec{k}.$$

(1b) Total angular momentum in field:

$$\hat{\vec{J}} = \int d^3x \ \vec{x} \times \hat{\vec{P}}(\vec{x})$$

where $\hat{\vec{P}}(\vec{x}) = \frac{1}{4\pi c} (\hat{\vec{E}} \times \hat{\vec{B}})$ = momentum density

Lets massage these equations a bit.

$$\begin{aligned} \cdot (\hat{\vec{E}} \times \hat{\vec{B}})_i &= \epsilon_{ijk} E_j B_k \quad (\text{summation convention}) \\ &= \epsilon_{ijk} E_j \epsilon_{kem} \partial_e A_m \\ &= (\delta_{ie}\delta_{jm} - \delta_{im}\delta_{je}) E_j \partial_e A_m \\ &= E_\ell \partial_i A_\ell - E_\ell \partial_\ell A_i \end{aligned}$$

Now $\overline{\vec{J}} = \int d^3x \ \vec{x} \times \overline{\vec{P}}(\vec{x})$

$$\begin{aligned} \Rightarrow \vec{J}_j &= \epsilon_{jki} \int d^3x \ x_k P_i \\ &= \epsilon_{jki} \frac{1}{4\pi c} \int d^3x \left[E_\ell (x_k \partial_i) A_\ell - (x_k E_\ell) (\partial_\ell A_i) \right] \\ &= \frac{1}{4\pi c} \int d^3x \ E_\ell (\vec{x} \times \vec{\nabla})_j A_\ell \\ &\quad + \frac{1}{4\pi c} \int d^3x \underbrace{\epsilon_{jki} \partial_\ell (x_k E_\ell)}_{[\delta_{ik} + \vec{\nabla} \times \vec{E}]} A_i \quad (\text{integration by parts}) \end{aligned}$$

\circ in free space

$$\Rightarrow \vec{J}_j = \frac{1}{4\pi c} \left(\int d^3x E_\epsilon (\vec{x} \times \vec{\nabla})_j A_\epsilon + \int d^3x (\vec{E} \times \vec{A})_j \right)$$

$$\Rightarrow \vec{J} = \vec{J}_{\text{orb}} + \vec{J}_{\text{spin}}$$

$\vec{J}_{\text{orb}} = \frac{1}{4\pi c} \int d^3x E_\epsilon (\vec{x} \times \vec{\nabla}) A_\epsilon$
$\vec{J}_{\text{spin}} = \frac{1}{4\pi c} \int d^3x (\vec{E} \times \vec{A})$

(1c)

Let us expand these terms in the plane wave basis:

$$\begin{aligned} \vec{J}_{\text{orb}} &= \left(\frac{1}{4\pi c} \int d^3x E_\epsilon^{(-)} (\vec{x} \times \vec{\nabla}) A_\epsilon^{(+)} + \text{k.c.} \right) \\ &\quad + \left(\frac{1}{4\pi c} \int d^3x E_\epsilon^{(+)} (\vec{x} \times \vec{\nabla}) \vec{A}_\epsilon^{(+)} + \text{k.c.} \right) \end{aligned}$$

Consider

$$\begin{aligned} \frac{1}{4\pi c} \int d^3x E_\epsilon^{(-)} (\vec{x} \times \vec{\nabla}) A_\epsilon^{(+)} \\ = \sum_{\vec{k}, \lambda, \lambda'} \frac{\hbar}{2} \sqrt{\frac{\omega'_\epsilon}{\omega_k}} \hat{a}_{\vec{k}\lambda'}^\dagger \hat{a}_{\vec{k}\lambda}^\dagger \vec{E}_{\vec{k}\lambda}^* \cdot \vec{E}_{\vec{k}\lambda'} \\ \underbrace{\int \frac{d^3x}{V} e^{-i\vec{k}' \cdot \vec{x}} (\vec{x} \times -i\vec{\nabla}) e^{i\vec{k} \cdot \vec{x}}}_d \end{aligned}$$

(Summing over ℓ)

$$\begin{aligned}
 \text{Aside: } \mathcal{J} &= \int \frac{d^3x}{V} e^{-i\vec{k} \cdot \vec{x}} (\vec{x} \times i\vec{\nabla}) e^{i\vec{k} \cdot \vec{x}} \\
 &= \int \frac{d^3x}{V} e^{-i\vec{k}' \cdot \vec{x}} (\vec{x} \times \vec{k}) e^{i\vec{k} \cdot \vec{x}} \\
 &= \left[\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}') \cdot \vec{x}} \right] \times \vec{k} \\
 &= +i\vec{\nabla}_{\vec{k}'} \left[\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}') \cdot \vec{x}} \right] \times \vec{k} \\
 \Rightarrow &\frac{1}{4\pi c} \int d^3x \sum_{\ell} \left(\vec{x} \times \vec{\nabla} \right) \hat{A}_{\ell}^{(+)} \\
 &= \sum_{\substack{\vec{k}, \vec{k}', \lambda, \lambda' \\ \lambda, \lambda'}} \frac{1}{2} \sqrt{\frac{\omega_k}{\omega_{k'}}} \hat{a}_{\vec{k}' \lambda'}^\dagger \hat{a}_{\vec{k}, \lambda}^\dagger \vec{\epsilon}_{\vec{k}, \lambda}^* \cdot \vec{\epsilon}_{\vec{k}' \lambda'} \\
 &\quad + i\vec{\nabla}_{\vec{k}'} \delta_{(\vec{k}, \vec{k}')} \times \vec{k}
 \end{aligned}$$

Though this is a finite sum, we are ultimately interested in the limit $V \rightarrow \infty$, when the sum goes to an integral and we can perform integration by parts.

$$\Rightarrow = \frac{1}{2} \sum_{\vec{k}, \lambda} \left(+i\vec{\nabla}_{\vec{k}} \hat{a}_{\vec{k}, \lambda}^\dagger \right)^+ \times \vec{k} \hat{a}_{\vec{k}, \lambda}$$

$$\text{having used } \vec{\epsilon}_{\vec{k}, \lambda} \cdot \vec{\epsilon}_{\vec{k}', \lambda'}^* = \delta_{\lambda \lambda'}$$

after integrating over \vec{k}' with the delta func.

Adding the conjugate term: $E_{\ell}^{(+)}(\vec{x} \times \vec{\nabla}) A_{\ell}^{(-)}$

$$\vec{J}_{\text{orbital}} = \frac{1}{2} \sum_{\vec{k}, \lambda} \left[(i \vec{\nabla}_{\vec{k}} \hat{a}_{\vec{k}, \lambda})^+ \times \hbar \vec{k} \hat{a}_{\vec{k}, \lambda} + \hbar \vec{k} \hat{a}_{\vec{k}, \lambda}^+ \times (i \vec{\nabla}_{\vec{k}} \hat{a}_{\vec{k}, \lambda}) \right]$$

$$= \sum_{\vec{k}, \lambda} (i \vec{\nabla}_{\vec{k}} \hat{a}_{\vec{k}, \lambda})^+ \times \hbar \vec{k} \hat{a}_{\vec{k}, \lambda}$$

$$+ \underbrace{\sum_{\vec{k}, \lambda} \cancel{\hbar \vec{k}} i \vec{\nabla}_{\vec{k}} (\hat{a}_{\vec{k}, \lambda}^+ \hat{a}_{\vec{k}, \lambda})}_{\text{Integration by parts} \Rightarrow 0}$$

Integration by parts $\Rightarrow 0$

$$\Rightarrow \vec{J}_{\text{orbital}} = \sum_{\vec{k}, \lambda} (i \vec{\nabla}_{\vec{k}} \hat{a}_{\vec{k}, \lambda})^+ \times \hbar \vec{k} \hat{a}_{\vec{k}, \lambda}$$

or equivalent

$$= \sum_{\vec{k}, \vec{k}', \lambda} \hat{a}_{\vec{k}', \lambda}^+ [i \vec{\nabla}_{\vec{k}'}, \delta(\vec{k} - \vec{k}') \times \hbar \vec{k}] \hat{a}_{\vec{k}, \lambda}$$

This is the "second quantized" form of orbital angular momentum. Recall from wave mechanics

$$\vec{L}_{\text{orbital}} = \vec{x} \times \hat{p} \doteq [i \vec{\nabla}_{\vec{k}}, \delta(\vec{k} - \vec{k}')] \times \hbar \vec{k}$$

in momentum space? Thus, if we have an electromagnetic wave packet (beam/pulse) we generally carry both orbital and spin angular momentum.

Consider now the spin term

$$\int d^3x \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} = -\frac{i\hbar}{2} \sum_{\vec{k}, \vec{k}', \lambda, \lambda'} \hat{a}_{\vec{k}\lambda}^\dagger \hat{a}_{\vec{k}'\lambda'} \vec{e}_{\vec{k}, \lambda}^* \times \vec{e}_{\vec{k}', \lambda'}^*$$

$$\underbrace{\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}') \cdot \vec{x}}}_{\delta^{(3)}(\vec{k}-\vec{k}')} \\ = -\frac{i\hbar}{2} \sum_{\vec{k}, \vec{k}'} \left[(\vec{e}_{\vec{k},+}^* \times \vec{e}_{\vec{k},+}) \hat{a}_{\vec{k}+}^\dagger \hat{a}_{\vec{k}+} + (\vec{e}_{\vec{k},-}^* \times \vec{e}_{\vec{k},-}) \hat{a}_{\vec{k}-}^\dagger \hat{a}_{\vec{k}-} \right]$$

As above $\vec{e}_{\vec{k},\pm} = \frac{\vec{e}_1 \pm i\vec{e}_2}{\sqrt{2}}$

where \vec{e}_1 and \vec{e}_2 are two orthonormal vectors with $\vec{e}_1 \times \vec{e}_2 = \hat{h}$

$$\Rightarrow \vec{e}_{\vec{k},+}^* \times \vec{e}_{\vec{k},+} = \pm \vec{e}_{\vec{k}}$$

U $\int d^3x \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} = \frac{\hbar}{2} \sum_{\vec{k}} (\hat{a}_{\vec{k}+}^\dagger \hat{a}_{\vec{k}+} - \hat{a}_{\vec{k}-}^\dagger \hat{a}_{\vec{k}-}) \vec{e}_{\vec{k}}$

Now $\int d^3x \frac{\vec{E}^{(+)} \times \vec{A}^{(-)}}{4\pi c} = \frac{\hbar}{2} \sum_{\vec{k}} (\hat{a}_{\vec{k}+} \hat{a}_{\vec{k}+}^\dagger - \hat{a}_{\vec{k}-} \hat{a}_{\vec{k}-}^\dagger) \vec{e}_{\vec{k}}$
 $= \frac{\hbar}{2} \sum_{\vec{k}} (\hat{a}_{\vec{k}+}^\dagger \hat{a}_{\vec{k}+} - \hat{a}_{\vec{k}-}^\dagger \hat{a}_{\vec{k}-}) \vec{e}_{\vec{k}}$ (commutation cancels)

Finally note: $\vec{e}_{\vec{k},\pm} \times \vec{e}_{\vec{k},\pm} = 0$

$$\Rightarrow \int d^3x \vec{E}^{(+)} \times \vec{A}^{(+)} = \int d^3x \vec{E}^{(-)} \times \vec{A}^{(-)} \\ = 0$$

Thus

$$\vec{J}_{\text{spin}} = \hbar \sum_{\vec{k}} (\hat{a}_{\vec{k},+}^\dagger \hat{a}_{\vec{k},+} - \hat{a}_{\vec{k},-}^\dagger \hat{a}_{\vec{k},-}) \hat{\epsilon}_{\vec{k}}$$

Each photon has intrinsic "spin" angular momentum. In the circularly polarized, plane wave basis, the photon has a definite helicity, ~~carries~~ carry one hbar of angular momentum along (opposite to) the direction of propagation $\hat{\epsilon}_{\vec{k}}$ for positive (negative) handed polarization.

The photon is spin $S=1$, yet there are only two states with definite projection of angular momentum, whereas, we might expect three ($2S+1 = 3$). This is a very subtle point coming from the fact the the photon is massless. For more details see,

"Photons and Atoms",

(1d) Mapping photon spin onto a two-state Hilbert space

$$\text{Define } \hat{\vec{J}}_{\text{spin}} = \hat{J}_x \hat{e}_x + \hat{J}_y \hat{e}_y + \hat{J}_z \hat{e}_z$$

$$\text{where } \hat{J}_x = \frac{\hbar}{2} (\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+)$$

$$\hat{J}_y = \frac{\hbar}{2i} (\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+)$$

$$\hat{J}_z = \frac{\hbar}{2} (\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)$$

This is the Schwinger representation of angular momentum connecting the "Boson algebra" $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ to the angular momentum algebra $[\hat{J}_i, \hat{J}_j] = i\hbar \epsilon_{ijk} \hat{J}_k$

$$\begin{aligned} \text{Check: } [\hat{J}_x, \hat{J}_y] &= \frac{\hbar^2}{4i} \left([\hat{a}_+^\dagger \hat{a}_-, -\hat{a}_-^\dagger \hat{a}_+] + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_-] \right) \\ &= \frac{\hbar^2}{4i} \left[\hat{a}_+^\dagger \hat{a}_+ ([\hat{a}_-^\dagger, \hat{a}_-]) - \underbrace{2\hat{a}_-^\dagger \hat{a}_-}_{=-1} (\underbrace{[\hat{a}_+^\dagger, \hat{a}_+]}_{=-1}) \right] \\ &= i\hbar \left(\frac{\hbar}{2} (\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-) \right) = i\hbar \hat{J}_z \quad \checkmark \end{aligned}$$

$$\begin{aligned} [\hat{J}_x, \hat{J}_z] &= \frac{\hbar^2}{4} \left([\hat{a}_+^\dagger \hat{a}_-, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_+^\dagger \hat{a}_-, \hat{a}_-^\dagger \hat{a}_-] \right. \\ &\quad \left. + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_-^\dagger \hat{a}_-] \right) \\ &= \frac{\hbar^2}{4} \left(\hat{a}_+^\dagger \hat{a}_- (-1) - \hat{a}_+^\dagger \hat{a}_- (1) + \hat{a}_-^\dagger \hat{a}_+ (1) - \hat{a}_-^\dagger \hat{a}_+ (-1) \right) \\ &= -\frac{\hbar^2}{2} (\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+) = -i\hbar \hat{J}_y \quad \checkmark \end{aligned}$$

$$\begin{aligned}
[\hat{J}_y, \hat{J}_z] &= \frac{\hbar^2}{4i} \left([\hat{a}_+^\dagger \hat{a}_-, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_+^\dagger \hat{a}_-, \hat{a}_-^\dagger \hat{a}_-] \right. \\
&\quad \left. - [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_+] + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_-^\dagger \hat{a}_-] \right) \\
&= \frac{\hbar^2}{4i} (\hat{a}_+^\dagger \hat{a}_- (-) - \hat{a}_+^\dagger \hat{a}_- (1) - \hat{a}_-^\dagger \hat{a}_+ (1) + \hat{a}_-^\dagger \hat{a}_+ (1)) \\
&= -\frac{\hbar^2}{4i} (\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+) \\
&= i\hbar \left[\frac{\hbar}{2} (\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+) \right] = i\hbar \hat{J}_x \quad \checkmark
\end{aligned}$$

The Schwinger representation is the "second quantized form" of the spin $\frac{1}{2}$ operators

$$\begin{aligned}
\hat{J}_x &= \frac{\hbar}{2} (|+\downarrow\rangle\langle-\downarrow| + |-\downarrow\rangle\langle+\downarrow|) \\
\hat{J}_y &= \frac{\hbar}{2i} (|+\downarrow\rangle\langle-\downarrow| - |-\downarrow\rangle\langle+\downarrow|) \\
\hat{J}_z &= \frac{\hbar}{2} (|+\downarrow\rangle\langle+\downarrow| - |-\downarrow\rangle\langle-\downarrow|)
\end{aligned}$$

"Second quantize" $|+\downarrow\rangle \Rightarrow \hat{a}_+^\dagger$ create spin up or down
 $\langle+\downarrow| \Rightarrow \hat{a}_+$ annihilate spin up or down

Thus, we can easily map the spin angular momentum of the photon onto the Bloch sphere, also

known as the Poincaré sphere as we visited in P5#1.