Physics 566: Quantum Optics I
Problem Set 7 Extra Credit
Due: Friday, November 1, 2019

Problem 1: Momentum and Angular Momentum in the E&M Field (25 points)

From classical electromagnetic field theory we know that conservation laws require that the field carry momentum and angular momentum

$$\mathbf{P} = \int d^3x \left(\frac{\mathbf{E}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})}{4\pi c} \right), \quad \mathbf{J} = \int d^3x \left(\mathbf{x} \times \frac{\mathbf{E}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})}{4\pi c} \right).$$

(a) Show that when these quantities become field operators, the momentum operator becomes, $\hat{\mathbf{P}} = \sum_{\mathbf{k},\lambda} \hbar \mathbf{k}\, \hat{a}^\dagger_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda}$; interpret.

(b) Show that $\mathbf{J} = \mathbf{J}_{orb} + \mathbf{J}_{spin}$

where $\mathbf{J}_{orb} = \dfrac{1}{4\pi c} \displaystyle\int d^3x\, E_i(\mathbf{x}) (\mathbf{x} \times \nabla) A_i(\mathbf{x})$, $\mathbf{J}_{spin} = \dfrac{1}{4\pi c} \displaystyle\int d^3x \left(\mathbf{E}(\mathbf{x}) \times \mathbf{A}(\mathbf{x}) \right)$

(c) Show that

$\hat{\mathbf{J}}_{orb} = \sum_{\mathbf{k}\,\mathbf{k}'} \sum_{\lambda} \hat{a}^\dagger_{\mathbf{k}'\lambda} \left(i\hbar \nabla_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{k}') \times \mathbf{k} \right) \hat{a}_{\mathbf{k}\lambda}$, where $\nabla_{\mathbf{k}}$ is the gradient in **k-space**, and

$\hat{\mathbf{J}}_{spin} = \hbar \sum_{\mathbf{k}} \left(\hat{a}^\dagger_{\mathbf{k},+} \hat{a}_{\mathbf{k},+} - \hat{a}^\dagger_{\mathbf{k},-} \hat{a}_{\mathbf{k},-} \right) \mathbf{e}_{\mathbf{k}}$. Interpret these quantities.

(d) The spin of the photon has magnitude S=1, yet there are only two helicity states. Thus we can map the spin angular momentum onto the Bloch(Poincaré) sphere for S=1/2, via

$$\hat{\mathbf{J}}_{spin} = \hat{J}_x \mathbf{e}_x + \hat{J}_y \mathbf{e}_y + \hat{J}_z \mathbf{e}_z,$$

$$\text{with } J_z = \frac{\hbar}{2} (\hat{a}^\dagger_{z+} \hat{a}_{z+} - \hat{a}^\dagger_{z-} \hat{a}_{z-}), \quad J_x = \frac{\hbar}{2} (\hat{a}^\dagger_{z+} \hat{a}_{z-} + \hat{a}^\dagger_{z-} \hat{a}_{z+}), \quad J_y = \frac{\hbar}{2i} (\hat{a}^\dagger_{z+} \hat{a}_{z-} - \hat{a}^\dagger_{z-} \hat{a}_{z+}),$$

where $(\hat{a}_{z+}, \hat{a}_{z-})$ are the mode operators for positive and negative helicity operators relative to a *space fixed* quantization axis.

(di) Show that these operators satisfy the SU(2) commutation algebra for angular momentum. This relationship is know as the "Schwinger representation" (see Sakauri).

(dii) The mean values of $\hat{J}_x, \hat{J}_y, \hat{J}_z$ are the "Stokes parameters" in classical optics and the Bloch vector components on the Poincaré sphere. Explain the relationship between these operators and the Pauli operators.