Physics 566: Quantum Optics I
Problem Set 3
Due Thursday, September 19, 2019

Problem 1: Different ensemble decompositions - example, spin ½ (15 points)

(a) Suppose we have a statistical mixture of spin 1/2 particles that consists of the state $|\uparrow_z\rangle$ mixture with probability $\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)$ and the state $|\downarrow_z\rangle$ with probability $\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)$.

Find the matrix of the density operator in the basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$, and in the basis of eigenstates of $\hat{\sigma}_x$, $\{|\uparrow_x\rangle, |\downarrow_x\rangle\}$. What is the Bloch vector that describes this state?

(b) Now suppose we have an mixed state with 1/2 probability to have spin along $\mathbf{e}_{n_1} = \frac{1}{\sqrt{2}}(\mathbf{e}_z + \mathbf{e}_x)$ and 1/2 probability to have spin along $\mathbf{e}_{n_2} = \frac{1}{\sqrt{2}}(\mathbf{e}_z - \mathbf{e}_x)$. Is this a completely mixed state? Write the density operator in the basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$. Compare to part (b). Please comment on your result.

(c) Show that two statistical mixtures of pure states, $\{|\uparrow_n\rangle\}$ with probabilities p_n, and $\{|\uparrow_m\rangle\}$ with probabilities q_m, describe the *same* density operator $\hat{\rho}$ if

$$\mathbf{Q} = \sum_n p_n \mathbf{e}_n = \sum_m q_m \mathbf{e}_m ,$$

where $\mathbf{Q}$ is the Bloch vector of $\hat{\rho}$. Check this with your results of parts (b) and (c).

Problem 2: Ambiguity of ensemble decompositions of density operators (15 points)

We saw in Problem 1 that a density operator does not decompose uniquely into a statistical mixture of pure states. This has profound implications for both practical calculations of the density matrix (as we will see later in the semester) as well as for foundational descriptions of states in quantum mechanics. What different ensemble are possible to yield a given density operator? In this problem we prove the following.

<u>Hughston-Jozsa-Wootters (HJW) theorem:</u> The two density operators

$$\hat{\rho}_1 = \sum_i p_i |\psi_i\rangle\langle\psi_i| \text{ and } \hat{\rho}_2 = \sum_j q_j |\phi_j\rangle\langle\phi_j|$$

are equal if and only if the two ensembles are related by,

$$\sqrt{q_j}|\phi_j\rangle = \sum_i U_{ji}\sqrt{p_i}|\psi_i\rangle ,$$

where U_{ji} are elements of a unitary matrix.

(a) Assume the relation between the ensembles is true. Prove that $\hat{\rho}_1 = \hat{\rho}_2$.

(b) Assume $\hat{\rho}_1 = \hat{\rho}_2 \equiv \hat{\rho}$. Show $\sqrt{q_j}\left|\phi_j\right\rangle = \sum_i U_{ji}\sqrt{p_i}\left|\psi_i\right\rangle$.

(Hint: Show first that $\sqrt{p_i}\left|\psi_i\right\rangle = \sum_\alpha M_{j\alpha}\sqrt{\lambda_\alpha}\left|e_\alpha\right\rangle$, where λ_α are the eigenvalues of $\hat{\rho}$ and $\left|e_\alpha\right\rangle$ its orthonormal eigenvectors and $M_{j\alpha}$ are elements of a unitary matrix. The same thus holds for $\sqrt{q_j}\left|\phi_j\right\rangle$. The proof will follow).

Historical note: "The theorem was originally proven by Schrödinger in 1936. He commented that this theorem was one 'for which I claim no priority but the permission of deducing it in the following section, for it is certainly not well known.' His comment was amusingly prescient: The theorem was rediscovered by Jaynes in 1957 (whose work was extended by Hadjisavvas (1981)), rediscovered by Hughston, Jozsa, and Wootters (HJW) in 1993 (this last an expansion of a 1989 partial rediscovery by Gisin); in 1999, Mermin simplified a portion of HJW's proof - and it would appear none of these were aware of Schrödinger's work. Furthering the irony, Mermin commented that this is 'a pertinent theorem which deserves to be more widely known.' "

Problem 3: Qubits encoded in photon polarization (25 points)

The two orthogonal polarization states of a photon define a qubit. Let us define the standard basis

$$\frac{\mathbf{e}_H + i\mathbf{e}_V}{\sqrt{2}} : \text{right hand circular (positive helicity)} \Rightarrow \left|\uparrow_z\right\rangle$$

$$\frac{\mathbf{e}_H - i\mathbf{e}_V}{\sqrt{2}} : \text{left hand circular (negative helicity)} \Rightarrow \left|\downarrow_z\right\rangle$$

where $(\mathbf{e}_H, \mathbf{e}_V)$ are linear polarizations along some defined "horizontal" and "vertical" axes.

The Bloch sphere description of the polarization is known as the "Poincaré sphere", with each point on the surface representing a possible elliptical polarization. The three Cartesian coordinates of the Bloch vector are also known as the "Stokes parameters".

(a) To what polarization vectors do you associate $\left|\uparrow_x\right\rangle, \left|\downarrow_x\right\rangle$ and $\left|\uparrow_y\right\rangle, \left|\downarrow_y\right\rangle$? What is the nature of the polarization, linear, circular, elliptical, and along what direction if linear?

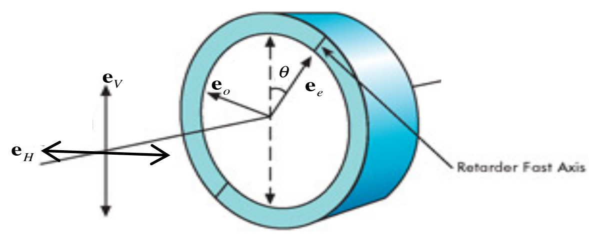
(b) Generalize: What is the polarization vector corresponding to an arbitrary state of the qubit, $\left|\uparrow_n\right\rangle$? Give the ration of the semi-major/minor axes of the ellipse in terms of the direction (θ,ϕ) of the state $\left|\uparrow_n\right\rangle$ on the Poincaré sphere? (Hint: See Jackson, Classical Electrodynamics on Stokes vector)

(c) Sketch the Poincaré sphere, denoting the polarization states at the north and south pole, and at a few points along the equator as well as a few great circle a with constant latitude/longitude.

A wave plate is an optical element with birefringence, i.e., the index of refraction is in different along two orthogonal axes, “ordinary” and “extraordinary” m n_o, n_e. The result is that the phase shift imparted to the light depends on the polarization of the light with eigenvectors

$$\mathbf{e}_o \Rightarrow e^{i\frac{\omega}{c}n_o L}\mathbf{e}_o, \quad \mathbf{e}_e \Rightarrow e^{i\frac{\omega}{c}n_e L}\mathbf{e}_e$$

where L is the thickness of crystal. By orienting the crystal at an angle θ with respect to the H,V axes, one can transform the polarization state.



(d) Show that transformation of the polarization state by the waveplate can be expressed an SU(2) rotation on the Poincaré sphere (neglecting the overall phase) in terms of the parameters θ and $\Delta\phi = 2\pi\frac{(n_e - n_0)L}{\lambda}$ as

$$\hat{U}_\theta^{WP}(\Delta\phi) = \hat{D}_3(2\theta)\hat{D}_1(\Delta\phi)\hat{D}_3^\dagger(2\theta) = \begin{bmatrix} \cos\left(\frac{\Delta\phi}{2}\right) + i\cos 2\theta \sin\left(\frac{\Delta\phi}{2}\right) & -i\sin 2\theta \sin\left(\frac{\Delta\phi}{2}\right) \\ -i\sin 2\theta \sin\left(\frac{\Delta\phi}{2}\right) & \cos\left(\frac{\Delta\phi}{2}\right) - i\cos 2\theta \sin\left(\frac{\Delta\phi}{2}\right) \end{bmatrix}$$

where $\hat{D}_i(\Theta) = e^{-i\frac{\Theta}{2}\hat{\sigma}_i}$ is the rotation operator around the i^{th} axis of the Poincaré sphere and the matrix is written in the basis $\{|\uparrow_x\rangle, |\downarrow_x\rangle\}$

(e) A quarter-wave plate has $L = \frac{\lambda}{4(n_e - n_o)}$; a half-wave plate has $L = \frac{\lambda}{2(n_e - n_o)}$.

How should the quarter-wave plate be oriented to transform horizontal polarization to circular polarization? What is the rotation on the Poincaré sphere?

How should a half-wave plate be oriented to transform horizontal polarization to vertical polarization? What is the rotation on the Poincaré sphere?

(f) Extra credit (5 points). Show that an arbitrary SU(2) transformation on the Poincaré sphere can be constructed using two quarter-wave plates and one half-wave plates.