Physics 566: Quantum Optics I
Problem Set 2
Due Thursday, September 12, 2019

Problem 1: Chaotic Light (25 Points)
Natural light arising from, e.g. stars, is not "coherent" in contrast to the light from a laser. The phase of the field fluctuates, and is only well correlated for a short "coherence time." One source of those fluctuations is random collisions between the radiators. In this problem you will study the nature of this coherence to set the stage for our study of filed coherence later in the semester.

(a) Let $P_s(t)$ be the "survival probability," i.e., the probability that the molecule freely oscillates and survives a time *t* without a collision. Under the assumption that the time of the next collision is independent of the previous (such as random process is said to be *Markovian* – there is no "memory" of the previous trajectory), show that

$P_s(t) = e^{-\gamma t}$, where $\gamma = 1/\tau_0$ is the rate of collisions, and τ_0 is the average time between collisions.

(b) Show that the probability that the oscillator free oscillates for time *t* and then suffers a collision between times *t* and *t+dt*

$$p(t)dt = e^{-t/\tau_0}\frac{dt}{\tau_0}.$$

Use the kinetic theory of gases to show that $1/\tau_0 = n\sigma_0\bar{v}_{rel}$, where *n* is the density of molecules, σ_0 is the collision cross section and $\bar{v}_{rel}$ is the average relative speed of the molecules. What is $\bar{v}_{rel}$ for a gas in thermal equilibrium?

(c) The electric field produced by each of the oscillators will has a random phase, $E_i(t) = E_0 e^{-i\omega t}e^{i\phi_i(t)}$. The total field $E(t) = \sum_{i=1}^{N} E_0 e^{-i\omega t}e^{i\phi_i(t)} = E_0 e^{-i\omega t}\alpha(t) = E_0 e^{-i\omega t}a(t)e^{i\varphi(t)}$, where $\alpha(t)$ is the random complex amplitude, $a(t) = |\alpha(t)|$ and φ is overall the random phase. Argue that for *N* large, the probability of a given complex amplitude is Gaussian distributed in amplitude, and independent of phase

$$p(\alpha(t)) = \frac{1}{\pi N}e^{-|\alpha(t)|^2/N}$$

(d) Argue that under the *ergodic assumption* (the random signal samples different values according to the given probability distribution, so ensemble averages equal time averages), the two-time correlation function is $\langle E^*(t)E(t+\tau)\rangle = \Gamma(\tau) = NE_0^2 e^{-i\omega\tau}e^{-\tau/\tau_0}$

Hint: Justify, and then use the fact that $\langle e^{i(\phi_i(t+\tau)-\phi_j(t))}\rangle = 0$ unless *i*=*j* and also that there is no collision between t and $t+\tau$.

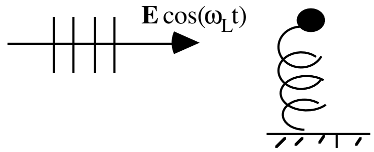
(e) The electric field, on average is zero, but there are fluctuations around the average. This implies that the intensity $I(t) = |E(t)|^2$ fluctuates. Using $E(t) = \sum_{i=1}^{N} E_0 e^{-i\omega t}e^{i\phi_i(t)}$, show that

$\langle I(t)\rangle = NE_0^2$, $\langle (I(t))^2\rangle = 2\langle I\rangle^2 \Rightarrow \Delta I = \langle I\rangle$, and generally the probability distribution of intensities is

$$P[I(t)] = \frac{1}{\langle I\rangle}e^{-\frac{I(t)}{\langle I\rangle}} \Rightarrow \langle (I(t))^n\rangle = n!\langle I\rangle^n$$

Problem 2: Lorentz oscillator model of scattering (20 points)

Consider the scattering of an electromagnetic wave by a damped Lorentz oscillator



(a) The absorption cross section, σ_{abs}, is defined as the rate at which energy is absorbed by an atom, divided by the incident flux of energy, the intensity $I = \frac{c}{8\pi}|\mathbf{E}_0|^2$ (CGS units). Show that the classical model of absorption gives,

$$\sigma_{abs,class} = \frac{2\pi^2 e^2}{mc} g(\omega_L), \text{ where } g(\omega) = \frac{\Gamma_{rad}/(2\pi)}{(\omega-\omega_0)^2 + \Gamma_{rad}^2/4} \text{ is the line shape.}$$

Assume near resonance so that $\Delta = \omega_L - \omega_0 << \omega_L, \omega_0$.

(b) In the case of radiative damping, all energy absorbed is re-radiated, and is thus *scattered*. Use standard scattering theory to derive the differential scattering cross section for the Lorentz oscillator model, $\frac{d\sigma_{scat}}{d\Omega}$, and after integrating over all solid angles, show that the total scattering cross section equals the absorption cross section found in part (a). Here take $\Gamma_{rad} = \frac{2}{3}\frac{e^2}{mc^3}\omega_0^2$.

(c) We can re-derive the expression for the classical natural linewidth Γ_{rad} that we found in class via radiation reaction by looking directly at energy conservation in the scattering process. For the field on resonance, equate the time averaged absorbed power (rate at which field does work on electron, averaged over a period of oscillation) to the Larmor formula for the averaged radiated power to show,

$$\Gamma_{rad} = \frac{2}{3}\frac{e^2}{mc^3}\omega_0^{\,2} = \frac{2}{3}\left(k_0 r_c\right)\omega_0, \text{ where } r_c \text{ is the classical electron radius.}$$

Evaluate this for the case the sodium “D2 resonance” (the yellow light in stree light), of excitation wavelength is 589 nm. The quantum decay rate is $\Gamma/2\pi = 9.8$ MHz. What is the oscillator strength of the transition?

(d) Show that the scattering cross section can be rëexpressed as $\sigma_{scat} = \frac{6\pi\lambda\!\!\!^{-}{}_0^2}{1+\left(4\Delta^2/\Gamma_{rad}^2\right)}$, where $\lambda\!\!\!^{-}{}_0 = \lambda/2\pi$. This expression holds true quantum mechanically as well with $\Gamma_{rad} \to \Gamma$.