

Physics 566 : Quantum Optics I

Problem Set #1: Solutions

Problem 1: Gaussian Probability Distributions

For one random variable $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\langle x \rangle)^2}{2\sigma^2}}$

(a) The characteristic function $\chi(k) = \int_{-\infty}^{\infty} dx e^{ikx} p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx e^{-\frac{(x-\langle x \rangle)^2}{2\sigma^2}} e^{ikx}$

$$\Rightarrow \chi(k) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dy e^{-\frac{y^2}{2\sigma^2}} e^{ik(y+\langle x \rangle)} = \underbrace{\frac{e^{ik\langle x \rangle}}{\sqrt{2\pi\sigma^2}}}_{\text{completing the square}} \underbrace{\int_{-\infty}^{\infty} dy e^{-\frac{(y-i\sigma^2 k)^2}{2\sigma^2}}}_{= \sqrt{2\pi\sigma^2}} e^{-\frac{k^2\sigma^2}{2}}$$

$$\Rightarrow \boxed{\chi(k) = e^{ik\langle x \rangle} e^{-\frac{k^2\sigma^2}{2}}}$$

(b) Let $y = x - \langle x \rangle$. Then $P(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}$ and the corresponding characteristic function $\chi_y(k) = e^{-\frac{k^2\sigma^2}{2}}$. The moments are related to the Taylor series expansion

$$\chi_y(k) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle y^n \rangle = \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \langle y^n \rangle k^n = e^{-\frac{k^2\sigma^2}{2}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\sigma^{2m}}{2^m} k^{2m}$$

Now since $\chi_y(k)$ is an even function of k , all odd terms in the Taylor series vanish $\langle y^n \rangle = \langle (x - \langle x \rangle)^n \rangle = 0$ n odd

$$\Rightarrow \chi_y(k) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \langle y^{2m} \rangle k^{2m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\sigma^{2m}}{2^m} k^{2m}$$

$$\Rightarrow \langle y^{2m} \rangle = \frac{(2m)!}{m! 2^m} \sigma^{2m} = (2m-1)!! \sigma^{2m} \quad (\text{using Gamma function})$$

$$\langle (x - \langle x \rangle)^n \rangle = (n-1)!! \sigma^n = (n-1)(n-3)\cdots 1 \sigma^n \quad n \text{ even}$$

E.g. $\langle (x - \langle x \rangle)^2 \rangle = \sigma^2$, $\langle (x - \langle x \rangle)^4 \rangle = 3\sigma^4$, $\langle (x - \langle x \rangle)^6 \rangle = 15\sigma^6$

(c) The multivariate Gaussian, joint probability distribution

$$p(\vec{x}) = \frac{1}{\sqrt{(2\pi)^N \det C}} \exp \left[-\frac{1}{2} (\vec{x} - \vec{\alpha})^T \cdot C^{-1} \cdot (\vec{x} - \vec{\alpha}) \right]$$

\uparrow
 $C = \text{covariance matrix}$

The characteristic function $\chi(\vec{k}) = e^{i\vec{k} \cdot \vec{\alpha}} \exp \left\{ -\frac{1}{2} \vec{k}^T C \cdot \vec{k} \right\}$, using the same techniques as we used in part (a)

$$\Rightarrow \langle \vec{x} \rangle = -i \nabla_{\vec{k}} \chi(\vec{k}) \Big|_{\vec{k}=0} = \vec{\alpha}$$

$$\begin{aligned} \Rightarrow \langle x_i x_j \rangle &= -\partial_{k_j} \partial_{k_i} \chi(\vec{k}) \Big|_{\vec{k}=0} = -\partial_{k_j} \left[\left(i \alpha_i - \sum_l C_{il} k_l \right) e^{i\vec{k} \cdot \vec{\alpha}} \exp \left\{ -\frac{1}{2} \vec{k}^T C \cdot \vec{k} \right\} \right] \Big|_{\vec{k}=0} \\ &= \alpha_i \alpha_j + C_{ij} = \langle x_i \rangle \langle x_j \rangle + C_{ij} \end{aligned}$$

$$\Rightarrow \langle \Delta x_i \Delta x_j \rangle = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle = C_{ij} \quad \checkmark$$

(d) Now take $\langle \vec{x} \rangle = 0$, $\chi(\vec{k}) = \exp \left\{ -\frac{1}{2} \vec{k}^T C \cdot \vec{k} \right\}$, $\langle x_i x_j \rangle = C_{ij}$

$$\underbrace{\langle x_i x_j \cdots x_m \rangle}_{m \text{ terms}} = (-i)^m \frac{\partial^m}{\partial x_i \partial x_j \cdots \partial x_m} \exp \left\{ -\frac{1}{2} \sum_{ij} k_i C_{ij} k_j \right\} \Big|_{\vec{k}=0} = 0 \quad m \text{ odd}$$

Consider thus $m=2n$ (even)

$$\begin{aligned} \text{Example: } \langle x_1 x_2 x_3 x_4 \rangle &= \frac{\partial^4}{\partial x_1 \partial x_2 \partial x_3 \partial x_4} \chi(\vec{k}) = (-i)^4 \partial_4 \partial_3 \partial_2 \partial_1 \left(-\sum_l C_{1l} k_l e^{-\frac{1}{2} \vec{k}^T C \cdot \vec{k}} \right) \Big|_{\vec{k}=0} \\ &= \partial_4 \partial_3 \left\{ -C_{12} + \left(\sum_l C_{1l} k_l \right) \left(\sum_p C_{2p} k_p \right) \right\} e^{-\frac{1}{2} \vec{k}^T C \cdot \vec{k}} \Big|_{\vec{k}=0} \\ &= \partial_4 \left\{ C_{12} \sum_q C_{3q} k_q + C_{13} \sum_p C_{2p} k_p + \sum_l C_{1l} k_l C_{23} \right\} e^{-\frac{1}{2} \vec{k}^T C \cdot \vec{k}} \Big|_{\vec{k}=0} \\ &= C_{12} C_{34} + C_{13} C_{24} + C_{14} C_{23} \end{aligned}$$

$$\Rightarrow \langle x_1 x_2 x_3 x_4 \rangle = \langle x_1 x_2 \rangle \langle x_3 x_4 \rangle + \langle x_1 x_3 \rangle \langle x_2 x_4 \rangle + \langle x_1 x_4 \rangle \langle x_2 x_3 \rangle$$

This generalizes to the sum of all partitions of the $2n$ into pairs. There are $(2n-1)!!$ terms. Thus as expected $\langle x_i^{2n} \rangle = (2n-1)!! \langle x_i^2 \rangle$.

Problem 2: Wiener - Khinchin Theorem

(a) We begin with a function whose Fourier transform exists (i.e. a square normalizable function). $\tilde{f}(\omega) \equiv \int dt f(t) e^{i\omega t}$, $f(t) = \int \frac{d\omega}{2\pi} \tilde{f}(\omega) e^{-i\omega t}$ (take real)

Autocorrelation function over all time: $C(\tau) = \int_{-\infty}^{\infty} dt f(t) f(t+\tau) dt$

This is nothing but the convolution of f with itself. It then follows

$$C(\tau) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \tilde{f}(\omega) \tilde{f}(\omega') e^{-i\omega t} e^{-i\omega'(t+\tau)} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \tilde{f}(\omega) \tilde{f}^*(\omega') \underbrace{\int dt e^{-i(\omega+\omega')t}}_{2\pi \delta(\omega+\omega')} e^{+i\omega'\tau}$$

$$\Rightarrow C(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{f}(\omega) \tilde{f}(-\omega) e^{+i\omega\tau} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{f}(\omega) \tilde{f}^*(\omega) e^{-i\omega\tau} \text{ when } f(t) \text{ is real}$$

$$\Rightarrow C(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\tilde{f}(\omega)|^2 e^{+i\omega\tau} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\tilde{f}(-\omega)|^2 e^{-i\omega\tau} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\tilde{f}(\omega)|^2 e^{-i\omega\tau}$$

inverse $|\tilde{f}(\omega)|^2 = \int_{-\infty}^{\infty} d\tau C(\tau) e^{+i\omega\tau}$

(b) For a stationary process, the spectral density (or power spectral density) is defined in terms of the time averaged power per frequency interval.

$$S(\omega) \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \left| \int_{-T/2}^{T/2} f(t) e^{-i\omega t} dt \right|^2$$

The autocorrelation function is the time-average $G(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t) f(t+\tau) dt$

$$\text{Let } f_T(t) = \Theta_T(t) f(t) = \begin{cases} f(t) & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \Rightarrow S(\omega) = \lim_{T \rightarrow \infty} \frac{1}{T} |\tilde{f}_T(\omega)|^2$$

$$\begin{aligned} \Rightarrow \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S(\omega) e^{-i\omega\tau} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\tilde{f}_T(\omega)|^2 e^{-i\omega\tau} = \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} dt f_T(t) f_T(t+\tau) \quad (\text{as in part a}) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt f(t) f_T(t+\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t) f(t+\tau) \Theta_T(t+\tau) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t) f(t+\tau) dt \quad \text{since } \Theta_T(t+\tau) = 1 \text{ in integrand} \\ &\quad \text{as } T \rightarrow \infty \end{aligned}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S(\omega) e^{-i\omega\tau} = G(\tau), \quad S(\omega) = \int d\tau G(\tau) e^{+i\omega\tau}$$

(c) Let $f(t)$ be an ergodic, stationary process

$$\text{Then } G(\tau) = \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} f(t) f(t+\tau) dt = \underbrace{\langle f(t) f(t+\tau) \rangle}_{\substack{\uparrow \text{ergodic}}} = \underbrace{\langle f(0) f(\tau) \rangle}_{\substack{\uparrow \text{stationary}}} \leftarrow \begin{array}{l} \text{ensemble} \\ \text{average} \end{array}$$

$$\begin{aligned} \langle \tilde{f}_1^*(\omega) \tilde{f}_2(\omega') \rangle &= \left\langle \left(\int_{-\infty}^{\infty} dt f(t) e^{i\omega t} \right)^* \left(\int_{-\infty}^{\infty} dt' f(t') e^{i\omega' t'} \right) \right\rangle \\ &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' e^{i(\omega' t' - \omega t)} \underbrace{\langle f(t) f(t') \rangle}_{G(t'-t) \text{ (stationary)}} \end{aligned}$$

$$\text{Change variables: } \begin{aligned} \tau &= t' - t \\ T &= \frac{t + t'}{2}, \quad \text{Jacobi:an } dt_1 dt_2 = dt' dt \end{aligned}$$

$$\Rightarrow \langle \tilde{f}_1^*(\omega) \tilde{f}_2(\omega') \rangle = \int_{-\infty}^{\infty} d\tau e^{i(\omega' - \omega)\tau} \int_{-\infty}^{\infty} d\tau e^{i(\omega + \omega')\frac{\tau}{2}} G(\tau) = 2\pi \delta(\omega - \omega') \int_{-\infty}^{\infty} e^{i\omega\tau} G(\tau) d\tau$$

$$\Rightarrow \langle \tilde{f}_1^*(\omega) \tilde{f}_2(\omega') \rangle = 2\pi S(\omega) \delta(\omega - \omega'), \quad S(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle f(0) f(\tau) \rangle \equiv \text{Spectral density}$$

(d) Fourier transform of the real signal $\tilde{E}_R(\omega) = \int_{-\infty}^{\infty} dt E(t) e^{-i\omega t}$

$$E(t) = \underbrace{\int_0^{\infty} \frac{d\omega}{2\pi} \tilde{E}(\omega) e^{-i\omega t}}_{E^{(+)}(t)} + \underbrace{\int_0^{\infty} \frac{d\omega}{2\pi} \tilde{E}^*(\omega) e^{+i\omega t}}_{E^{(-)}(t)}$$

$$\text{Analytic signal: } \tilde{E}_c(t) = 2\tilde{E}_c^{(+)}(t) = 2 \int_0^{\infty} \frac{d\omega}{2\pi} \tilde{E}(\omega) e^{-i\omega t}, \quad E(t) \equiv \text{Re}(\tilde{E}(t)) = \frac{1}{2} (\tilde{E}(t) + \tilde{E}^*(t))$$

Complex correlation function $\Gamma(\tau) = \langle E^{(+)}(0) E^{(+)}(\tau) \rangle$ (for stationary / ergodic process)

$$\begin{aligned} \Gamma(\tau) &= \int_0^{\infty} \frac{d\omega}{2\pi} \int_0^{\infty} \frac{d\omega'}{2\pi} \underbrace{\langle \tilde{E}(\omega) \tilde{E}^*(\omega') \rangle}_{2\pi \delta(\omega - \omega') S(\omega)} e^{-i\omega'\tau} = \int_0^{\infty} S(\omega) e^{-i\omega\tau} \frac{d\omega}{2\pi} \\ &\quad S(\omega) \end{aligned}$$

$$\Gamma(\tau) + \Gamma^*(\tau) = 2 \text{Re}(\Gamma(\tau)) = \int_0^{\infty} S(\omega) e^{-i\omega\tau} \frac{d\omega}{2\pi} + \int_0^{\infty} S^*(\omega) e^{+i\omega\tau} \frac{d\omega}{2\pi}$$

$S(-\omega)$ since $G(\tau)$ real

$$\Rightarrow \text{Re}(\Gamma(\tau)) = \frac{1}{2} \int_{-\infty}^{\infty} S(\omega) e^{-i\omega\tau} \frac{d\omega}{2\pi}, \quad S(\omega) = 2 \int_{-\infty}^{\infty} d\tau \text{Re}(\Gamma(\tau)) e^{+i\omega\tau}$$

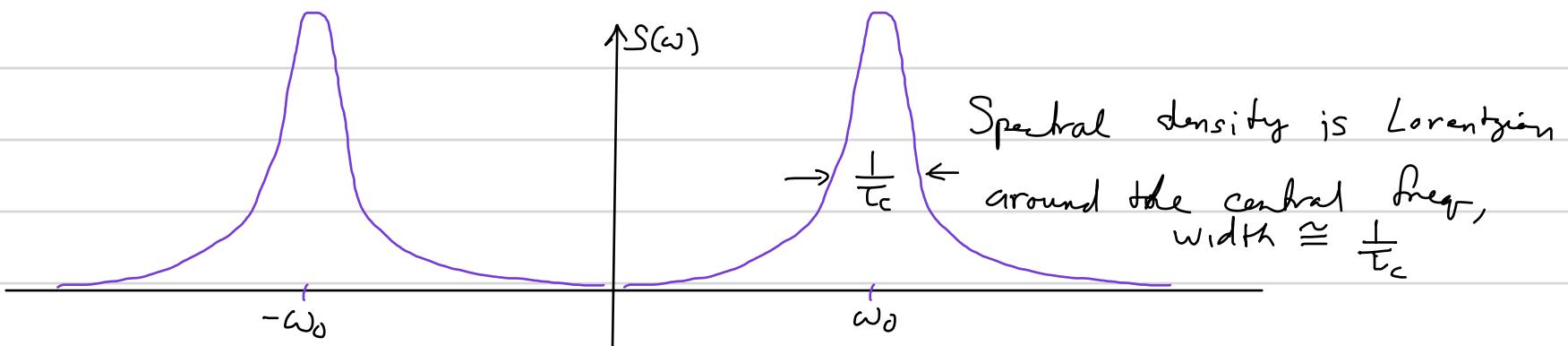
(c) The power spectrum is defined by the spectral density $S(\omega)$. Consider "natural light," e.g. light from a thermal lamp, collision broadened. We found in class, the autocorrelation function between the complex amplitudes

$$\langle E^{(1)}(\tau_0) E^{(2)*}(\tau) \rangle = \frac{I_0}{4} e^{-i\omega_0 \tau - \frac{|\tau|}{\tau_c}}, \quad \text{where } \omega_0 = \text{central frequency}, \quad \frac{1}{\tau_c} = \frac{\text{Collision rate}}{I_0 = \text{intensity at } \tau=0}$$

$$\Rightarrow \text{Spectral density: } S(\omega) = \frac{I_0}{2} \int_{-\infty}^{\infty} d\tau \cos(\omega_0 \tau) e^{-\frac{|\tau|}{\tau_c}} e^{+i\omega \tau} = \frac{I_0}{4} \int_{-\infty}^{\infty} d\tau [e^{+i(\omega - \omega_0)\tau - \frac{|\tau|}{\tau_c}} + e^{i(\omega + \omega_0)\tau - \frac{|\tau|}{\tau_c}}]$$

$$\begin{aligned} \text{Aside: } \int_{-\infty}^{\infty} d\tau e^{i\Omega\tau - \frac{|\tau|}{\tau_c}} &= \int_0^{\infty} d\tau e^{(-i\Omega - \frac{1}{\tau_c})\tau} + \int_{-\infty}^0 d\tau e^{(-i\Omega + \frac{1}{\tau_c})\tau} \\ &= \frac{1}{-i\Omega - \frac{1}{\tau_c}} + \frac{-1}{-i\Omega + \frac{1}{\tau_c}} = \frac{2/\tau_c}{\Omega^2 + (\frac{1}{\tau_c})^2} \end{aligned}$$

$$\Rightarrow S(\omega) = \frac{I_0}{2} \left[\frac{1/\tau_c}{(\omega - \omega_0)^2 + (1/\tau_c)^2} + \frac{1/\tau_c}{(\omega + \omega_0)^2 + (1/\tau_c)^2} \right]$$



The output intensity of the interferometer is the Fourier transform of the spectral density \Rightarrow For a Lorentzian power spectral density, fringes decay exponentially.

