Physics 566: Quantum Optics I
Problem Set 1
Due Thursday, September 5, 2019

Problem 1: Gaussian probability distributions (20 points)

Consider a Gaussian for one random variable $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\langle x\rangle)^2}{2\sigma^2}}$.

(a) Show that the characteristic function is $\chi(k) = e^{ik\langle x\rangle - \frac{k^2\sigma^2}{2}}$

(b) Show that $\left\langle \left(x-\langle x\rangle\right)^n \right\rangle = \begin{cases} 0, & n\ odd \\ (n-1)!!\sigma^n, & n\ even \end{cases}$

Now consider a mutinomial Gaussian for a the random variables $\mathbf{x} = (x_1, x_2, ..., x_N)$,

$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi N)\det C}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{a})^T \cdot C^{-1} \cdot (\mathbf{x}-\mathbf{a})\right]$, where C is the "covariance matrix," an *NXN* symmetric matrix with nonnegative eigenvalues.

(c) Show that $\langle \mathbf{x} \rangle = \mathbf{a}$, $\langle \Delta x_i \Delta x_j \rangle = \left\langle \left(x_i - \langle x_i\rangle\right)\left(x_j - \langle x_j \rangle\right)\right\rangle = C_{ij}$

(d) A Gaussian is completely specified by its mean can covariance matrix. For simplicity let take $\langle \mathbf{x} \rangle = 0$. Show that all the moments of the multivariate Gaussian are defined in terms of the two point-correlations

$$\langle x_1 x_2 \cdots x_{2n} \rangle = \sum\prod \langle x_i x_j \rangle = \sum\prod C_{ij}$$

$$\langle x_1 x_2 \cdots x_{2n-1} \rangle = 0$$

where the notation $\sum\prod$ means summing over all distinct ways of partitioning $x_1, \ldots, x_n$ into pairs x_i, x_j and each summand is the product of the n pairs. For example

$$\langle x_1 x_2 x_3 x_4 \rangle = \langle x_1 x_2 \rangle\langle x_3 x_4 \rangle + \langle x_1 x_3 \rangle\langle x_2 x_4 \rangle + \langle x_1 x_4 \rangle\langle x_2 x_3 \rangle$$

Problem 2: Wiener-Khinchin Theorem (25 Points)

Consider a real function $f(t)$ (this could be a deterministic or random process). Defining the Fourier transform in our usual way, $\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{+i\omega t}\, dt$; this exists if $f(t)$ is square integrable. In this case according to Parceval's theorem $\int_{-\infty}^{\infty} f^2(t)\,dt = \int_{-\infty}^{\infty} \left|\tilde{f}(\omega)\right|^2 \frac{d\omega}{2\pi}$ and $\left|\tilde{f}(\omega)\right|^2$ is known as the spectral density.

(a) Let $C(\tau)=\int_{-\infty}^{\infty} f(t)f(t+\tau)\,dt$ (autocorrelation function). Show that for $f(t)$ real

$$\left|\tilde{f}(\omega)\right|^2=\int_{-\infty}^{\infty} C(\tau)e^{i\omega\tau}\,d\tau\,,\quad C(\tau)=\int_{-\infty}^{\infty}\left|\tilde{f}(\omega)\right|^2 e^{-i\omega\tau}\frac{d\omega}{2\pi}$$

That is, the spectrum density is the Fourier transform of the autocorrelation function and vice versa. This a form of the *Wiener-Khinchin Theorem*.

For a formally stationary process, $\tilde{f}(\omega)$ does not exist. In that case we have to be a little more careful. One defines the time average power $P=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f^2(t)\,dt=\int_{-\infty}^{\infty} S(\omega)\frac{d\omega}{2\pi}$, where $S(\omega)$ is the power spectral density. If follow that $S(\omega)=\lim_{T\to\infty}\frac{1}{T}\left|\int_{-T/2}^{T/2} f(t)e^{i\omega t}\,dt\right|^2$

(b) Show that the general form of the Wiener-Khinchin Theorem is

$$S(\omega)=\int G(\tau)e^{+i\omega\tau}d\tau\ \text{, where }\ G(\tau)\equiv\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f(t)f(t+\tau)\,dt=\int S(\omega)e^{-i\omega\tau}\frac{d\omega}{2\pi}$$

(c) Now let $f(t)$ be an *ergodic and stationary* random process. Show that

$\left\langle \tilde{f}^*(\omega)\tilde{f}(\omega')\right\rangle=2\pi S(\omega)\delta(\omega-\omega')$, where angle brackets is the ensemble average.

Next, note that for a real function $\tilde{f}(-\omega)=\tilde{f}^*(\omega)$, Thus

$$f(t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{f}(\omega)e^{-i\omega t}=\underbrace{\int_{0}^{\infty}\frac{d\omega}{2\pi}\tilde{f}(\omega)e^{-i\omega t}}_{f^{(+)}(t)}+\underbrace{\int_{0}^{\infty}\frac{d\omega}{2\pi}\tilde{f}^*(\omega)e^{+i\omega t}}_{f^{-}(t)}$$

$f^{(\pm)}(t)$ is known as the "positive/negative frequency component, and" for a real function $f^{(+)}(t)=\left[f^{-}(t)\right]^*$. Note, we often define the complex "analytic signal" $\tilde{f}_c(t)=2f^{(+)}(t)$. Then $f(t)=\mathrm{Re}\left[\tilde{f}_c(t)\right]$.

(d) Consider the complex correlation function that determines temporal coherence in a standard interferometer for an ergodic, stationary process, $\Gamma(\tau)=\left\langle E^{(-)}(0)E^{(+)}(\tau)\right\rangle$. Show that

$$\mathrm{Re}\left[\Gamma(\tau)\right]=\frac{1}{2}\int_{-\infty}^{\infty} S(\omega)e^{-i\omega\tau}\frac{d\omega}{2\pi},\quad S(\omega)=2\int_{-\infty}^{\infty}\mathrm{Re}\left[\Gamma(\tau)\right]e^{+i\omega\tau}\,d\tau$$

(e) Quasimonochromatic natural light arising, i.e., from a star or a light bulb is "collision broadened." The autocorrelation in this case, as we shall see, is $\Gamma(\tau)=\left\langle E^{(-)}(0)E^{(+)}(\tau)\right\rangle=\frac{I_0}{4}e^{-i\omega_0\tau-|\tau|/\tau_c}$, where ω_0 is the peak frequency and τ_c is the mean free time between collisions. What is the power spectrum of the light? Sketch $S(\omega)$.