III Quantum Theory of Light

A. Canonical Quantization in the Coulomb Gauge

A.a Coulomb Gauge Lagrangian

Here I want to review the salient points of quantization of the electromagnetic field à 'la Cohen-Tannoudj or Klauder.

The Fundamental Lagrangian density (Rationalized natural units)

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} = \tfrac{1}{2}(\vec{E}^2 - \vec{B}^2)$$

$$A^\mu = (\phi, \vec{A}) \quad J^\mu = (\rho, \vec{J}) \qquad \text{where } \vec{E} = -\dot{\vec{A}} + \vec{\nabla}\phi \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

Decomposing the fields into solenoidal and irrotational parts

$$\vec{A}_\perp = \left(\overleftrightarrow{1} - \frac{\vec{\nabla}\vec{\nabla}}{\nabla^2}\right)\cdot\vec{A} \qquad \vec{A}_\parallel = \frac{\vec{\nabla}\vec{\nabla}}{\nabla^2}\cdot\vec{A}$$

$$\vec{\nabla}\cdot\vec{A}_\perp = 0 \qquad \vec{\nabla}\times\vec{A}_\parallel = 0$$

and remembering that the relavent physics is in the Lagrangian (or Action) $\int d^3x\, \mathcal{L}$ so that $\vec{A}\cdot\vec{A} = \vec{A}_\perp\cdot\vec{A}_\perp + \vec{A}_\parallel\cdot\vec{A}_\parallel$ (cross terms vanish in integration) gives the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\left(\dot{\vec{A}}_\perp^{\,2} - (\vec{\nabla}\times\vec{A}_\perp)^2\right) + \vec{J}_\perp\cdot\vec{A}_\perp + \tfrac{1}{2}\left(\dot{\vec{A}}_\parallel^{\,2} + 2\dot{\vec{A}}_\parallel\cdot\vec{\nabla}\phi + (\vec{\nabla}\phi)^2\right) + \vec{J}_\parallel\cdot\vec{A}_\parallel - \rho\phi$$

The crucial point is that $\mathcal{L}$ is independent of $\dot{\phi}$. Thus, the conjugate field $\frac{\partial\mathcal{L}}{\partial\dot{\phi}}$ is identically zero,

and therefore ϕ is effectively a c-number rather than an operator since it commutes with all other fields. (Next Page)

In ~~general~~ canonical theory, when a "velocity" of a generalized coordinate does not appear in Lagrangian then this degree of freedom can be eliminated via the Euler-Lagrange equation

$$-\nabla^2\phi - \vec{\nabla}\cdot\dot{\vec{A}}_{\parallel} - \rho = 0 \quad \Rightarrow \phi = -\frac{1}{\nabla^2}\{\rho + \vec{\nabla}\cdot\dot{\vec{A}}_{\parallel}\}$$

where $\frac{-1}{\nabla^2}$ is the integral operator $\frac{-1}{\nabla^2}f(\vec{x}) = \int d^3k\, \frac{\tilde{f}(\vec{k})}{k^2} e^{ik}$

$$= \int d^3x' \frac{f(\vec{x}-\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}$$

Then

$$(\vec{E}_{\parallel})^2 = (\dot{\vec{A}}_{\parallel} + \vec{\nabla}\phi)^2 = (\dot{\vec{A}}_{\parallel} - \frac{\vec{\nabla}}{\nabla^2}\rho - \frac{\vec{\nabla}\vec{\nabla}}{\nabla^2}\cdot\dot{\vec{A}}_{\parallel})^2$$

$$= \Big[\underbrace{(\overleftrightarrow{1} - \frac{\vec{\nabla}\vec{\nabla}}{\nabla^2})\cdot\dot{\vec{A}}_{\parallel}}_{=0 \text{ by definition of } \parallel} - \frac{\vec{\nabla}}{\nabla^2}\rho\Big]^2 = \Big(\frac{\vec{\nabla}}{\nabla^2}\rho\Big)\cdot\Big(\frac{\vec{\nabla}}{\nabla^2}\rho\Big)$$

$$= -\rho\frac{1}{\nabla^2}\rho \quad \text{(Effective integration by parts)}$$

$$\rho\phi = -\rho\frac{1}{\nabla^2}\rho + \frac{\vec{\nabla}}{\nabla^2}\rho\cdot\dot{\vec{A}}_{\parallel}$$

Thus

$$\mathcal{L} = \tfrac{1}{2}(\dot{\vec{A}}_{\perp}^2 - (\vec{\nabla}\times\vec{A}_{\perp})^2) + \vec{J}_{\perp}\cdot\vec{A}_{\perp} + \tfrac{1}{2}\vec{E}_{\parallel}^2 - \rho\phi + \vec{J}_{\parallel}\cdot\vec{A}_{\parallel}$$

becomes

$$\mathcal{L} = \tfrac{1}{2}(\dot{\vec{A}}_{\perp}^2 - (\vec{\nabla}\times\vec{A}_{\perp})^2) + \vec{J}_{\perp}\cdot\vec{A}_{\perp} + \tfrac{1}{2}\rho\frac{1}{\nabla^2}\rho + \vec{J}_{\parallel}\cdot\vec{A}_{\parallel} - \frac{\vec{\nabla}}{\nabla^2}\rho\cdot\dot{\vec{A}}$$

Now, the equation of motion generated by $\vec{A}_{\parallel}$ is

$$\vec{J}_{\parallel} + \frac{\vec{\nabla}}{\nabla^2}\dot{\rho} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{J}_{\parallel} + \frac{\partial\rho}{\partial t} = 0$$

This is just conservation of charge already contained in another Maxwell equation, and completely independent of the value of $\vec{A}_{\parallel}$. (Next page)

Thus, we see that $\vec{A}_{\parallel}$ is not a dynamical variable. This fact is clear already from gauge invariance $\vec{A} \Rightarrow \vec{A} + \vec{\nabla}\chi$ leaves physical variables unchanged. Under a gauge change $\vec{A}_{\perp} \Rightarrow \vec{A}_{\perp}$ $\vec{A}_{\parallel} \Rightarrow \vec{A}_{\parallel} + \vec{\nabla}\chi$. Thus $\vec{A}_{\parallel}$ can be chosen arbitrarily.

Coulomb gauge: Choose $\vec{A}_{\parallel} = 0$ $\quad \vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{A}_{\perp} = 0$

Then $\nabla^2\phi = -\rho$ $\quad \phi = -\frac{1}{\nabla^2}\rho = \int d^3x' \frac{\rho(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}$

(Instaneous Coloumb response)

$$\Rightarrow \frac{1}{2}\vec{E}_{\parallel}^{2} - \rho\phi = +\frac{1}{2}\rho\frac{1}{\nabla^2}\rho = -\frac{1}{2}\int d^3x' \frac{\rho(\vec{x})\rho(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}$$

$= -U_{coul}$ = -static coulomb energy density

In the Colomb Gauge then $\vec{A} = \vec{A}_{\perp}$

$$\mathcal{L} = \frac{1}{2}\left(\dot{\vec{A}}_{\perp}^{2} - (\vec{\nabla}\times\vec{A}_{\perp})^2\right) + \vec{J}_{\perp}\cdot\vec{A}_{\perp} - U_{coul}$$

$$\Rightarrow \boxed{\mathcal{L} = \frac{1}{2}\left(\dot{\vec{A}}_{\perp}^{2} - (\nabla_i\vec{A})^2\right) + \vec{J}_{\perp}\cdot\vec{A}_{\perp} - U_{coul}}$$ (Integration by parts)

The equation of motion is the wave equation

$$\Box\vec{A}_{\perp} = \vec{J}_{\perp}$$

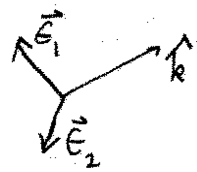
Note: the source is the solenoidal part of the current which is non-locally related to the total current.

A.b Solenoidal delta function tensor

$$\overleftrightarrow{\delta}^{S}(\vec{x}-\vec{x}') = \left(\overleftrightarrow{1} - \frac{\vec{\nabla}\vec{\nabla}}{\nabla^2}\right)\delta(\vec{x}-\vec{x}') = \int\frac{d^3k}{(2\pi)^3}\left(\overleftrightarrow{1} - \frac{\vec{k}\vec{k}}{k^2}\right)e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$\vec{\nabla}_{\vec{x}}\cdot\overleftrightarrow{\delta}^{S}(\vec{x}-\vec{x}') = \vec{\nabla}_{\vec{x}'}\cdot\overleftrightarrow{\delta}^{S}(\vec{x}-\vec{x}') = 0$$

In reciprical space ($\vec{k}$-space) we define the solenoidal basis: $\{\vec{\epsilon}_1(\vec{k}), \vec{\epsilon}_2(\vec{k}), \hat{k}\}$ $\quad \hat{k} = \frac{\vec{k}}{|\vec{k}|}$



$$\vec{\epsilon}_1 \times \vec{\epsilon}_2 = \hat{k} \qquad \left(\begin{array}{c}\vec{\epsilon} \text{ chosen} \\ \text{real here}\end{array}\right)$$

$$\hat{k}\cdot\vec{\epsilon}_1 = \hat{k}\cdot\vec{\epsilon}_2 = 0$$

The projection into the solenoidal plane (spanned by $\vec{\epsilon}_1$ & $\vec{\epsilon}_2$)

$$\overleftrightarrow{P}(\vec{k}) = \vec{\epsilon}_\lambda(\vec{k})\,\vec{\epsilon}_\lambda(\vec{k}) = \overleftrightarrow{1} - \hat{k}\hat{k} \qquad \text{(sum over } \lambda \text{ implied)}$$

Thus $\overleftrightarrow{\delta}^S(\vec{x}-\vec{x}') = \int \frac{d^3k}{(2\pi)^3}\, \overleftrightarrow{P}(\vec{k})\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$

Thus $\vec{A}_\perp(\vec{x}) = \int \vec{\epsilon}_\lambda(\vec{k})\, \mathcal{A}_\lambda(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$ where $\mathcal{A}_\lambda(\vec{k}) = \vec{\epsilon}_\lambda(\vec{k}) \cdot \int \frac{d^3k}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot x} \vec{A}_\perp(\vec{x})$

A.c Quantization

Starting with the Lagrangian boxed on page III.A.3

$$\vec{\Pi} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\vec{A}}} = \dot{\vec{A}} = -\vec{E} \qquad \left(\begin{array}{l}\text{Hence forth the solenoidal} \\ \text{subscript is understood}\end{array}\right)$$

The canonical commutation relation must be modified to account for the solenoidal constraint

$$[\vec{A}(\vec{x},t), \vec{\Pi}(\vec{x}',t)] = i\,\overleftrightarrow{\delta}^{A}(\vec{x}-\vec{x}') = [\vec{E}(\vec{x}',t), \vec{A}(\vec{x},t)]$$

$$[A_i(\vec{x},t), E_j(\vec{x}',t)] = -i\,\delta^{A}_{ij}(\vec{x}-\vec{x}')$$

A.d Creation/Annihillation Operators

Translating ~~into~~ the Lagrangian into $\vec{k}$-space we must keep in mind that the reality of the field implies

$$\mathcal{A}_\lambda(\vec{k})^* = \mathcal{A}_\lambda(-\vec{k})$$

Thus, only the field in the "positive half" $\vec{k}$-plane are dynamically independent.

Thus $L = \int_{k>0} d^3\vec{k} \left[\{ \dot{\mathcal{A}}_\lambda(\vec{k})\, \dot{\mathcal{A}}^*_\lambda(\vec{k}) - k^2 |\mathcal{A}_\lambda(\vec{k})|^2 \} + j^*_\lambda(\vec{k})\mathcal{A}_\lambda(\vec{k}) + j_\lambda(\vec{k})\mathcal{A}^*_\lambda(\vec{k}) \right.$

For <u>complex</u> fields $\Pi_\lambda(\vec{k}) = \frac{\partial \mathcal{L}}{\partial \dot{\mathcal{A}}^*_\lambda(\vec{k})} = \dot{\mathcal{A}}_\lambda(\vec{k})$

$$[\mathcal{A}_\lambda(\vec{k},t),\ \Pi^\dagger_{\lambda'}(\vec{k}',t)] = i\,\delta_{\lambda\lambda'}\,\delta(\vec{k}-\vec{k}')$$

The creation/annihillation operators are then defined

$$a_\lambda(\vec{k},t) = \frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(\omega_{\vec{k}}\,\mathcal{A}_\lambda(\vec{k},t) + i\,\Pi_\lambda(\vec{k},t)\right) \qquad (\omega_{\vec{k}} = |\vec{k}|)$$

$$\tfrac{1}{2}\left(\mathcal{A}_\lambda(\vec{k},t) + \mathcal{A}^\dagger_\lambda(-\vec{k},t)\right) = \frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_\lambda(\vec{k},t) + a^\dagger_\lambda(-\vec{k},t)\right) = \mathcal{A}_\lambda(\vec{k})\ (\forall \vec{k})$$

$$\tfrac{1}{2}\left(\Pi_\lambda(\vec{k},t) + \Pi^\dagger_\lambda(-\vec{k},t)\right) = -i\sqrt{\frac{\omega_{\vec{k}}}{2}}\left(a_\lambda(\vec{k},t) - a^\dagger_\lambda(-\vec{k},t)\right) = \Pi_\lambda(\vec{k})\ (\forall \vec{k})$$

$$[a_\lambda(\vec{k},t),\ a^\dagger_{\lambda'}(\vec{k}',t)] = \frac{-i}{2}[\mathcal{A}_\lambda(\vec{k},t),\ \Pi^\dagger_{\lambda'}(\vec{k}',t)] + \frac{i}{2}[\mathcal{A}^\dagger_{\lambda'}(\vec{k}',t),\ \Pi_\lambda(\vec{k},t)]$$

$$= \delta_{\lambda\lambda'}\,\delta(\vec{k}-\vec{k}') \quad \text{(Standard canonical commutation relations)}$$

Thus

$$\vec{A}(\vec{x},t) = \int\frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_\lambda(\vec{k},t) + a^\dagger_\lambda(-\vec{k},t)\right)e^{i\vec{k}\cdot\vec{x}}$$

$$= \int\frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_k}}\left(a_\lambda(\vec{k},t)\,e^{i\vec{k}\cdot\vec{x}} + a^\dagger_\lambda(\vec{k},t)\,e^{-i\vec{k}\cdot\vec{x}}\right)$$

For free fields $a_\lambda(\vec{k},t) = a_\lambda(\vec{k},0)\,e^{-i\omega_k t}$ $\cancel{= a_\lambda(\vec{k})\,e^{i\vec{k}\cdot\vec{x}-\omega_k}}$

Thus $\int\frac{d^3k}{(2\pi)}\,\frac{1}{\sqrt{2\omega_k}}\,a_\lambda(\vec{k},t)\,e^{i\vec{k}\cdot\vec{x}} = \vec{A}^{(+)}(\vec{x})$ the positive frequency component

$$\vec{A}^{(+)}(\vec{x},t) = \frac{1}{2}\left(\vec{A}(\vec{x},t) + \frac{i}{\sqrt{-\nabla^2}}\,\vec{\Pi}(\vec{x},t)\right)$$

= Absorptive part of the field
(positive freq component for free ~~fields~~ fields
= twice the analytic signal)

A.8 Hamiltonian and Heisenberg equations of motion

Performing the standard Lagendre transformation

$$\mathcal{H} = \vec{\Pi}\cdot\dot{\vec{A}} - \mathcal{L}(\vec{A}, \dot{\vec{A}}(\vec{\Pi}), \nabla_i \vec{A})$$

$$= \frac{1}{2}(\vec{\Pi}^2 + (\nabla_i \vec{A})^2) - \vec{J}\cdot\vec{A} + U_{coul}$$

Plugging in for $\vec{A}$ and $\vec{\Pi}$ in terms of $a^\dagger_\lambda(\vec{k})$ and $a_\lambda(\vec{k})$

$$\Rightarrow \mathcal{H} = \frac{\omega_{\vec{k}}}{2}(a^\dagger_\lambda(\vec{k})\,a_\lambda(\vec{k}) + a_\lambda(\vec{k})\,a^\dagger_\lambda(\vec{k})) - \frac{1}{\sqrt{2\omega}}\left[j^\dagger_\lambda(\vec{k})\,a_\lambda(\vec{k}) + j_\lambda(\vec{k})\,a^\dagger_\lambda(\vec{k})\right]$$

$$\mathcal{H} = \omega_{\vec{k}}\,(a^\dagger_\lambda(\vec{k})\,a_\lambda(\vec{k}) + \underset{\substack{\Uparrow \\ \text{zero' point}}}{\frac{1}{2}}) - \frac{1}{\sqrt{2\omega}}\left(j^\dagger_\lambda(\vec{k})\,a_\lambda(\vec{k}) + j_\lambda(\vec{k})\,a^\dagger_\lambda(\vec{k})\right)$$

where $j_\lambda(\vec{k}) = \vec{\epsilon}_\lambda(\vec{k})\cdot \int \frac{d^3k}{(2\pi)^{3/2}}\, e^{-i\vec{k}\cdot\vec{x}}\, \vec{J}(\vec{x})$

The equations of motion can be ~~other~~ obtain through Heisenberg eq. of motion, with Hamiltonian $H = \int d^3k\, \mathcal{H}$

$$\dot{a}_\lambda(\vec{k}) = \frac{1}{i}\,[a_\lambda(\vec{k}), H] = -i\omega_{\vec{k}}\, a_\lambda(\vec{k}) + \frac{i}{\sqrt{2\omega_{\vec{k}}}}\, j_\lambda(\vec{k})$$

In configuration space

$$\dot{\vec{A}}^{(+)}(\vec{x},t) = -i\sqrt{-\nabla^2}\,\vec{A}^{(+)}(\vec{x},t) + \frac{i}{2\sqrt{-\nabla^2}}\,\vec{J}(\vec{x},t)$$

This also follows from classical equations of motion

$$\dot{\vec{A}} = \frac{\partial \mathcal{H}}{\partial \vec{\Pi}} = \vec{\Pi} \qquad \dot{\vec{\Pi}} = -\frac{\partial \mathcal{H}}{\partial \vec{A}} - \nabla_i \frac{\partial \mathcal{H}}{\partial \nabla_i \vec{A}} = \vec{J} + \nabla^2 \vec{A}$$

$$\dot{\vec{A}}^{(+)} = \frac{1}{2}(\dot{\vec{A}} + \frac{i}{\sqrt{-\nabla^2}}\,\dot{\vec{\Pi}}) = \frac{1}{2}(\vec{\Pi} + \frac{i}{\sqrt{-\nabla^2}}(\vec{J} - (\sqrt{-\nabla^2})^2\vec{A}))$$

$$= -i\sqrt{-\nabla^2}\left(\frac{1}{2}\left[\vec{A} + \frac{i}{\sqrt{-\nabla^2}}\,\vec{\Pi}\right] + \frac{i}{2\sqrt{-\nabla^2}}\,\vec{J}\right) = -i\sqrt{-\nabla^2}\,\vec{A}^{(+)} + \frac{i}{2\sqrt{-\nabla^2}}\,\vec{J}$$

A.7 Integration of the Eq of motion and Propagation

The Heisenberg equations of motion can be formally integrated:

$$a_\lambda(\vec{k},t) = a_\lambda^{in}(\vec{k},t) + \frac{i}{\sqrt{2\omega}} \int_{-\infty}^{t} dt' \, e^{-i\omega(t-t')} j_\lambda(\vec{k},t')$$

Where $a_\lambda^{in}(\vec{k},t)$ is the homogenous solution $a_\lambda^{in}(\vec{k}) e^{-i\omega t}$ ~~IA the initial state is the va~~ representing the field in the remote past when the effect of the sources is negligable and thus represents the vacuum fluctuation contribution

Note that this is solution expected given the greens function for the wave equation

$$G_{Ret}(|\vec{x}-\vec{x}'|, t-t') = \frac{1}{4\pi|\vec{x}-\vec{x}'|} \delta(t'-t_{Ret})$$

where $t_{Ret} = t - |\vec{x}-\vec{x}'|$

$$\Box \vec{A} = \vec{J}$$

$$\Rightarrow \vec{A}(\vec{x},t) = \int d^3x' dt' \, G_{Ret}(|\vec{x}-\vec{x}'|, t-t') \, \vec{J}(\vec{x}',t') = \int d^3x' \frac{\vec{J}(\vec{x}',t_{Ret})}{4\pi|\vec{x}-\vec{x}'|}$$

(here $\vec{A}_{in}$ is set to zero)

Now $a_\lambda(\vec{k}) = \vec{\epsilon}_\lambda(\vec{k}) \cdot \tilde{\vec{A}}^{(+)}(\vec{k}) \sqrt{2\omega_k}$

$$\tilde{\vec{A}}(\vec{k},t) = \int \frac{d^3x}{(2\pi)^{3/2}} \vec{A}(\vec{x},t) e^{-i\vec{k}\cdot\vec{x}} = \int dt' \, \tilde{G}_{Ret}(\vec{k}, t-t') \, \tilde{\vec{J}}(\vec{k},t')$$

(Conv'd Thm)

where

$$\tilde{G}_{Ret}(\vec{k},\tau) = \int \frac{d^3y}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{y}} G_{Ret}(\vec{y}, t-t') = \int \frac{d^3y}{(2\pi)^{3/2}} \frac{e^{-i\vec{k}\cdot\vec{y}}}{4\pi|\vec{y}|} \delta(|\vec{y}|-\tau)$$

$$= \int \frac{y^2 dy \, d\mu}{(2\pi)^{3/2}} \frac{e^{-iky\mu}}{2y} \delta(y-\tau) = \int \frac{dy}{(2\pi)^{3/2}} \frac{\sin ky}{k} \delta(y-\tau)$$

$$= \frac{\sin k\tau}{(2\pi)^{3/2} k} = \overset{?}{\frac{1}{(2\pi)^{3/2}}} \left\{ \frac{-i e^{i\omega_k \tau}}{2\omega_k} + \frac{i}{2\omega_k} e^{-i\omega_k \tau} \right\}$$

Thus, the positive frequency component

$$\tilde{G}^{(+)}_{Ret}(\vec{k},\tau) = \frac{i}{2\omega_k} e^{-i\omega_k \tau} \qquad \tau > 0$$

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$$\therefore \quad \tilde{\vec{A}}^{(+)}(\vec{k},t) = \vec{A}^{(+)}_{in}(\vec{k}t) + \frac{i}{2\omega_k}\int_{-\infty}^{t} dt'\, e^{-i\omega_k(t-t')}\,\tilde{\vec{J}}(\vec{k},t')$$

$$a_\lambda(\vec{k},t) = \sqrt{2\omega_k}\;\vec{\epsilon}_\lambda(\vec{k})\cdot\tilde{\vec{A}}(\vec{k},t)$$

$$= a^{in}_\lambda(\vec{k},t) + \frac{i}{\sqrt{2\omega_k}}\int_{-\infty}^{t} dt'\, e^{-i\omega_k(t-t')}\, j_\lambda(\vec{k},t')$$

where $j_\lambda(\vec{k},t) = \vec{\epsilon}_\lambda(\vec{k})\cdot\tilde{\vec{J}}(\vec{k},t)$

This agrees with the result on the top of page III.A.7

The Green's function for $\dot{\vec{A}}^{(+)} = -i\sqrt{-\nabla^2}\,\vec{A}^{(+)} + \frac{i}{2\sqrt{-\nabla^2}}\vec{J}$

is $G^{(+)}_{Ret}(\vec{y},\tau) = \int d^3k\; e^{i\vec{k}\cdot\vec{y}}\;\tilde{G}^{(+)}_{Ret}(\vec{k},\tau)$

For spatial propagation, we consider the Helmhotz eq.

$$(\nabla^2 + \omega_k^2)\vec{A}(\vec{x},\omega_{\vec{k}}) = \vec{J}(\vec{x},\omega_{\vec{k}})$$

the Green function

$$G'_{Helm}(|\vec{x}-\vec{x}'|) = \int d\omega_k\; e^{i\omega_k\tau}\, G_{Ret}(|\vec{x}-\vec{x}|,\tau)$$

$$= \frac{1}{4\pi|\vec{x}-\vec{x}'|}\, e^{i\omega_k|\vec{x}-\vec{x}'|}$$

$$(\nabla^2 + \omega_k^2)\, G_{Helm}(|\vec{x}-\vec{x}'|) = -\,\delta(\vec{x}-\vec{x}')$$

In a source free region we can solve the propagation problem

$$\vec{A}(\vec{x},\omega_{\vec{k}}) = \int_R \{ G_H(|\vec{x}-\vec{x}'|)\,(\nabla^2_{x'}+\omega^2)\vec{A}(\vec{x}',\omega) - \vec{A}(\vec{x}',\omega)\,(\nabla^2_{x'}+\omega^2)\,G_H(|\vec{x}-\vec{x}'|)]\, d^3x'$$

$$= \int_R \vec{\nabla}_{x'}\cdot\left[G_H(\vec{x}-\vec{x}')\,\vec{\nabla}_{x'}\vec{A}(\vec{x}',\omega) - \vec{A}(\vec{x}',\omega)\,\vec{\nabla}_{x'}G_H(\vec{x}-\vec{x}')\right] d^3x'$$

$$= \int_S \left\{ G_H(\vec{x}-\vec{x}')\,\frac{\partial}{\partial n}\vec{A}(\vec{x}',\omega) - \vec{A}(\vec{x}',\omega)\,\frac{\partial}{\partial n}G_H(\vec{x}-\vec{x}')\right\} d^3x'$$

If the derivatives of $\vec{A}$ vanish on the surface

$$\vec{A}(\vec{x},t) = \int_S K(\vec{x},\vec{y},t-t')\;\vec{A}(\vec{y},t')\; dS\,dt'$$

$$K(\vec{x},y,t-t') = -\int \frac{\partial G_H(\vec{x}-\vec{y})}{\partial n}\, e^{-i\omega(t-t')}\, d\omega$$

III.B Atom – Quantized field intractions for Two Level atoms

III B.a Interaction Hamiltonian

The fundamental intraction of charges and electromagnetic fields is already given from the complete Lagrangian boxed on page III.A.3. For optical applications the multipole expansion is generally valid since wavelengths are large compared to atomic dimensions. To do this easily it is convenient to add to the ~~total~~ Lagrangian a total derivative

$$L \Rightarrow L + \frac{dF}{dt} \quad \text{where} \quad F = -\int d^3x\, \vec{P}\cdot\vec{A} \quad (\vec{\nabla}\cdot\vec{A}=0)$$
$$= -\int d^3x\, \vec{P}_\perp\cdot\vec{A}_\perp$$

Then $\mathcal{L} \Rightarrow \mathcal{L} - \frac{d}{dt}(\vec{P}_\perp\cdot\vec{A})$

$$= \tfrac{1}{2}(\dot{\vec{A}}^2 - (\nabla_i\vec{A})^2) + \underbrace{(\vec{J}-\dot{\vec{P}})}_{=\vec{J}_M = \vec{\nabla}\times\vec{M} = \text{Magnetization current}}\cdot\vec{A} - \vec{P}_\perp\cdot\dot{\vec{A}} - U_{coul}$$

(assuming no free charges)

$$= \tfrac{1}{2}(\dot{\vec{A}}^2 - (\nabla_i\vec{A})^2) + \underbrace{\vec{M}\cdot(\vec{\nabla}\times\vec{A})}_{\vec{M}\cdot\vec{B}} \underbrace{- \vec{P}_\perp\cdot\dot{\vec{A}}}_{\vec{P}_\perp\cdot\vec{E}} - U_{coul}$$

<u>Note</u>: The change in the Lagrangian is equivalent to a gauge transformation with a corresponding change of representation. Here the conjugate field

$$\vec{\Pi} = \frac{\partial\mathcal{L}'}{\partial\dot{\vec{A}}} = \dot{\vec{A}} - \vec{P}_\perp = -\vec{D}_\perp \quad (\text{displacement vector})$$

The gauge transformation is carried out by the unitary operator $U = \exp\{-i\int d^3x\, \vec{P}_\perp\cdot\vec{A}\}$

For the case of interest, the states will be restricted to photons whose wavelengths are consistent with the multipole expansion.

In the lowest order, the dipole approximation holds

$$\vec{P}_\perp^{(1)} = \vec{\mu}\,\delta(\vec{x} - \vec{R}_{atom}) \qquad \vec{\mu} = \text{electric dipole moment} = e \cdot \vec{x}$$

$$\vec{M}^{(1)} = 0$$

$$L_{int} = \int d^3x \left(\vec{M}(x) \cdot \vec{B}(\vec{x}) + \vec{P}(x) \cdot \vec{E}(\vec{x}) \right) = \vec{\mu} \cdot \vec{E}(\vec{R}_{atom})$$

The unitary gauge change is then $U = e^{-ie\vec{x} \cdot \vec{A}(\vec{R}_{atom})}$ as on page I.A.1

The interaction Hamiltonian $H_{int} = -L_{int} = -\vec{\mu} \cdot \vec{E}(\vec{R}_{atom})$ as before

Thus $H_{Total} = H_{atom} + H_{Field} + H_{int}$

We again restrict our attention to quasi-monochromatic field near resonance to two levels of an atom, so that we restrict our attention to upper level $|a\rangle$ and $|b\rangle$. Then

$$H_{Total} = \hbar\omega_a |a\rangle\langle a| + \hbar\omega_b |b\rangle\langle b| + \int d^3k\, \hbar\omega_k \left[a_\lambda^\dagger(\vec{k}) a_\lambda(\vec{k}) + \tfrac{1}{2} \right]$$

$$- \vec{\mu}_{ab} \cdot \vec{E}(\vec{R}_{atom}) |a\rangle\langle b| - \vec{\mu}_{ba} \cdot \vec{E}(atom) |b\rangle\langle a|$$

In the Heisenberg picture

$$\vec{E} = \int \frac{d^3k}{(2\pi)^{3/2}} + i\sqrt{\frac{\omega_k}{2}} \left(a_\lambda(\vec{k},t) e^{i\vec{k}\cdot\vec{x}} - a_\lambda^\dagger(\vec{k},t) e^{-i\vec{k}\cdot\vec{x}} \right) \vec{\epsilon}_\lambda(\vec{k})$$

For many problems in quantum optics it is more convient to deal with box normalization, with either periodic or "hard-wall" boundary condition (Going to standard CGS units)

Periodic: $$\vec{E}(\vec{x},t) = \sum_{\vec{k},\lambda} i\sqrt{\frac{2\pi\hbar\omega_k}{V}} \left[a_{\vec{k},\lambda}(t) e^{i\vec{k}\cdot\vec{x}} - a^\dagger_{\vec{k},\lambda} e^{i\vec{k}\cdot\vec{x}} \right] \vec{\epsilon}_\lambda(\vec{k})$$

Hard Wall $$\vec{E}(\vec{x},t) = \sum_{\vec{k},\lambda} \sqrt{\frac{4\pi\hbar\omega_k}{V}} \left(a_{\vec{k}}(t) + a^\dagger_{\vec{k},\lambda}(\vec{k}) \right) \vec{\epsilon}_\lambda \sin kz$$

Again using the spin ½ Pauli operators

$$H_{Total} = \frac{1}{2}\hbar\omega_{ab}\sigma_z + \sum_{k,\lambda} \hbar\omega_k a^{\dagger}_{k,\lambda} a_{k,\lambda} \quad \text{(Drop zero point)}$$

$$- \sum_{\vec{k},\lambda} \Big[\frac{\hbar}{2} g_{\vec{k},\lambda}\, \sigma_+ \left(a_{\vec{k},\lambda} e^{i\vec{k}\cdot\vec{R}_{atom}} - a^{\dagger}_{\vec{k},\lambda} e^{i\vec{k}\cdot\vec{R}_{atom}}\right)$$

$$- \frac{\hbar}{2} g^*_{\vec{k},\lambda}\, \sigma_- \left(a_{\vec{k},\lambda} e^{i\vec{k}\cdot\vec{R}_{atom}} - a^{\dagger}_{\vec{k},\lambda} e^{i\vec{k}\cdot\vec{R}_{atom}}\right)\Big]$$

Where $\frac{1}{2}\hbar g_{\vec{k},\lambda} = i\sqrt{\frac{2\pi\hbar\omega_k}{V}}\, \vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\mu}_{ab}$

$g_{\vec{k},\lambda}$ is the "quantum Rabi frequency"

$= \frac{2\vec{E}_0 \cdot \vec{\mu}_{ab}}{\hbar}$ where $\vec{E}_0 = i\sqrt{\frac{2\pi\hbar\omega_k}{V}}\, \vec{\epsilon}_\lambda(\vec{k})$ is the electric field of the "vacuum" photon

For the remainder we will take R_{atom} to be the orgin.

The quantum ~~equit~~ equivalent to the rotating wave approximation (RWA) is to neglect the "non-energy" conserving terms $\sigma_+ a^{\dagger}$ and $\sigma_- a$. This is most easily seen in the interaction representation. Terms $\sigma_+ a_k \Rightarrow \sigma_+ a_k e^{i(\omega_{ab}-\omega_k)t}$ $\sigma_- a^{\dagger}_k \Rightarrow \sigma_- a^{\dagger}_k e^{-i(\omega_{ab}-\omega_k)t}$ have resonant, slowly varying exponentials where as the others are antiresonant. these contribute to virtual transitions in higher order processes that we are not considering thus

$$\boxed{H_{Total} = \frac{1}{2}\hbar\omega_{ab}\sigma_z + \sum_{k,\lambda}\hbar\omega_k a^{\dagger}_{\vec{k},\lambda} a_{\vec{k},} - \sum_{\vec{k},\lambda}\frac{\hbar}{2}\left\{ g_{\vec{k},\lambda}\sigma_+ a_{\vec{k},\lambda} + g^*_{\vec{k},\lambda}\sigma_- a^{\dagger}_{\vec{k},\lambda}\right\}}$$

Quasi-monochromatic Field int with two level atom in electric dipole and RWA