appendix A

Using Simple Algebra

In chapter 1, we described mathematics as part of the language of physics. It is a compact way of expressing relationships between physical quantities that makes manipulations of these relationships much easier than if they were expressed in words and sentences. People who are conversant in mathematics are very comfortable in interpreting and using such relationships.

Many people are not comfortable using simple mathematics, and therefore mathematics is used sparingly in this textbook. Most of what is used is basic algebra, to which most college students have been introduced in high school. Often that introduction fails to produce a firm understanding of the principles underlying algebraic operations, though, leaving students with little confidence in their use. This appendix presents the basic concepts underlying simple algebraic manipulations and provides illustrations of their application.

Basic Concepts

Three simple but fundamental concepts form the basis for most algebraic manipulations. These concepts are the following:

Concept 1: The letters used in algebra represent numbers. Any operation performed with numbers (addition, subtraction, multiplication, division, and so on) can also be performed with these symbols.

In mathematics courses, the letters x and y are often used to represent unknown numbers, and other letters are used to represent constants or known numbers. In physics, specific letters are used to represent specific quantities: t for time, m for mass, d for distance, s for speed, and so on. They all represent numerical quantities, but some may be known and others may be initially unknown. The relationship $s = d/t$, for example, tells us that we can find the numerical value of speed by dividing a numerical value for distance by a numerical value of time. This relationship holds for any possible values of the distance and time.

Concept 2: If the same operation is performed on both sides of an equation, the equality expressed by that equation does not change.

This principle is the basis for all algebraic manipulations performed to express a relationship in different forms. For example, if we multiply both sides of the equation $s = d/t$ by the quantity t, the equality still holds. This operation yields

$$st = d\frac{t}{t} = d$$

since t/t equals 1. Performing this operation expresses the original equation in a new form: $d = st$ tells us that the distance is equal to the speed multiplied by time. We can multiply both sides of an equation by the same quantity, divide both sides by the same quantity, add or subtract the same quantity from both sides, and the equation is still valid. We can also square both sides of the equation or perform various other operations, but the operations just listed are those most commonly used.

Concept 3: When we solve an algebraic equation, we are merely rearranging the equation as just described so that the quantity we wish to know is expressed, by itself, on one side of the equation and everything else is on the other side of the equation.

In the paragraph illustrating concept 2, we solved the equation $s = d/t$ for the quantity d, thus expressing the distance in terms of the other two quantities, speed and time. If we wanted to express the time of travel in terms of the speed and the distance, we could divide both sides of the equation $d = st$ by the quantity s:

$$\frac{d}{s} = \frac{st}{s} = t\left(\frac{s}{s}\right) = t$$

or, $t = d/s$, the distance divided by the speed. We see that the original equation $s = d/t$ can be expressed in two other forms, $d = st$ and $t = d/s$, that restate the original equality in forms suitable for computing a specific quantity when the other two quantities are known, an extremely useful thing to be able to do.

Since the letters represent numbers (concept 1), we can always check the validity of the operations we perform by inventing numbers for the quantities and checking to see that the equalities still hold in the new form. For example, if in the original equation, $d = 6$ cm and $t = 2$ s, then

$$s = \frac{d}{t} = \frac{6\text{ cm}}{2\text{ s}} = 3\text{ cm/s}.$$

If we put the same numbers in the final equation, $t = d/s$, we have

$$t = \frac{d}{s} = \frac{6\text{ cm}}{3\text{ cm/s}} = 2\text{ s}$$

or 2 s = 2 s, which is obviously an equality.

These concepts are straightforward, and their application is not difficult once the basic ideas are grasped. A little practice, obtained by following the additional examples given below and performing the exercises at the end of this appendix, should help to build confidence in their use. For most people who have trouble with mathematics, lack of confidence is the fundamental problem. Often, they have never fully accepted the idea that letters can represent numbers, and the manipulations and rules of algebra therefore seem arbitrary and mysterious.

Other Examples

1. Solve the equation $a = b + c$ for the quantity c.

Solution: We seek an expression in which c is by itself on one side of the equation and the other two quantities are on the other side. This can be accomplished by subtracting the quantity b from both sides of the equation, since doing so will leave c by itself on the right side:

$$a - b = b + c - b = c.$$

Thus we see that $c = a - b$. (It does not matter which side of an equality is stated first—the equality is the same in either case.) By subtracting b from the right side of the original equation, c now stands by itself, so we have achieved the desired result.

2. Solve the equation $v = v_0 + at$ for the quantity t.

Solution: This is best done in two steps. The first step is to subtract the quantity v_0 from both sides of the equation to isolate the product at:

$$v - v_0 = v_0 + at - v_0 = at$$

$$at = v - v_0.$$

Then we divide both sides of this equation by a to get t by itself:

$$\frac{at}{a} = \frac{v - v_0}{a}$$

$$t = \frac{v - v_0}{a}.$$

If you can understand why each of these operations was performed (what was the motivation or objective?), you are well on your way to following the algebra used in this textbook.

3. Solve the equation $b = c + d/t$ for the quantity t.

Solution: Again, we first subtract the quantity c from both sides of the equation to isolate the term containing t:

$$b - c = c + \frac{d}{t} - c = \frac{d}{t}.$$

The quantity t is in the denominator, however, so we multiply both sides of the equation by t:

$$(b - c)t = \frac{d}{t}t = d.$$

Next, we divide both sides of the equation by $(b - c)$ to obtain t by itself on the left side of the equation:

$$\frac{(b - c)t}{b - c} = \frac{d}{b - c}$$

$$t = \frac{d}{b - c}.$$

Although this is a more complex example, each of these steps has a specific objective. The first step isolates the quantity d/t, the second step removes t from the denominator so that we can more readily solve for t, and the final step leaves t by itself. These objectives must be recognized to gain confidence in performing such operations yourself. Even people who are familiar with algebra often forget just what they are trying to accomplish, or they get careless in making sure that they are doing the same thing to both sides of an equation.

Exercises

(Answers to odd-numbered exercises are found in appendix D.)

1. Solve the equation $F = ma$ for the quantity a.
2. Solve the equation $PV = nRT$ for the quantity P.
3. Solve the equation $b = c + d$ for the quantity d.
4. Solve the equation $h = g - f$ for the quantity g.
5. Solve the equation $a = bc + d$ for the quantity d.
6. Solve the equation in exercise 5 for the quantity b.
7. Solve the equation $a = b(c - d)$ for the quantity b.
8. Solve the equation in exercise 7 for the quantity c. (Hint: First rewrite the equation as $a = bc - bd$, multiplying both terms inside the parentheses by b. This does not change the equality.)
9. Solve the equation $a + b = c - d$ for the quantity b.
10. Solve the equation in exercise 9 for the quantity c.
11. Solve the equation $b(a + c) = dt$ for the quantity b.
12. Solve the equation in exercise 11 for the quantity c.
13. Solve the equation $x = v_0t + \frac{1}{2}at^2$ for the quantity v_0.
14. Solve the equation in exercise 13 for the quantity a.

appendix B

Decimal Fractions, Percentages, and Scientific Notation

In physics and many other fields in which numbers are important, we usually express fractions as decimal fractions and often use percentages as a means of expressing fractions or ratios. Because we need at times to deal with very large and very small numbers, we also use a means of expressing these numbers involving powers of ten or *scientific notation* to avoid writing out all of the zeros. Although scientific notation is used sparingly in this book, there are times when its use is highly desirable, if not essential, so it is important that you understand its meaning. It is part of the language of science.

Decimal Fractions

Although most college-level students are familiar with decimal fractions and percentages, they are not always completely sure of their meaning. Fractions involve ratios or proportions, which are not well understood by many people. One of the benefits of taking a course in physics is that it can strengthen your ability to think in terms of ratios or proportions and to understand how they are described.

A decimal fraction is just a fraction for which the number in the denominator is some multiple of the number 10 (10, 100, 1000, and so forth), with the appropriate multiple indicated by the location of the decimal point. For example, if we start with the fraction ½ and divide 1 by 2 as the fraction indicates, a calculator will display the result as 0.5. The decimal point in front of the 5 is a shorthand notation for expressing the fraction $^{5}/_{10}$. The number 5 is half of 10, so the fraction $^{5}/_{10}$ is the same as the fraction ½ (one-half). In other words, the *ratio* of 5 to 10 is the same as the ratio of 1 to 2.

If the fraction ¾ is evaluated on a calculator by dividing 3 by 4, the calculator will express the result as 0.75, which is equivalent to the fraction $^{75}/_{100}$ or 75 hundredths. Thus, the first place or number after the decimal point represents tenths, the second place hundredths, the third place thousandths, and so on. The fraction 346 thousandths ($^{346}/_{1000}$) is expressed as 0.346, for example. We could read this as 3 tenths plus 4 hundredths plus 6 thousandths.

Decimal fractions are used very commonly, although we may not always stop to think about their meaning. In baseball, for example, we express a batter's hitting efficiency as a decimal fraction. A batter who has produced 35 hits in 100 official at-bats is said to be hitting 350. This is really $^{350}/_{1000}$ or 0.350, but the decimal point is often omitted. Most people understand that it should be there, however, and that we are merely expressing the fraction $^{35}/_{100}$ in decimal form and including three figures to the right of the decimal point.

What Are Percentages?

Another common way of expressing decimal fractions is to write them as percentages. The word *percent* means per one hundred, so a percentage is just a decimal fraction in which the denominator is 100. The fraction ½, for example, is $^{5}/_{10}$ or $^{50}/_{100}$ and can be expressed as 50%—it is 50 hundredths. The fraction ¾ is 0.75 or 75% (75 hundredths), and the fraction $^{346}/_{1000}$ is 0.346 or 34.6% (34.6 hundredths). Thus, moving the decimal place two places to the right converts a decimal fraction to a percentage and is equivalent to multiplying the fraction by 100.

The use of percentages is even more common than the direct use of decimal fractions. Interest rates and tax rates are usually expressed as percentages, for example, so we should all have some understanding of their meaning. An interest rate of 7% means that you will receive or pay $7 each year for every $100 that you have invested or borrowed, $^{7}/_{100}$ of the total amount. (We are ignoring here the possible effects of compounded interest.) A tax rate of 28% means that we will owe to the government $28 of every $100 that we earn (after deductions are subtracted). A percentage is always per one hundred, by definition.

Although it is easy enough to understand how a percentage is calculated (compute the decimal fraction and multiply by 100), having a good feeling for the proportions represented by different percentages is another matter. Pie charts, like the one shown in figure B.1, are often used to provide a visual representation of these proportions. The slices of the pie should have sizes in proportion to the percentages or fractions being represented. If the graphic artist does not understand this (as sometimes happens), the resulting pie chart may be very misleading.

The pie chart in figure B.1 represents the average monthly expenditures of someone who takes home \$2000 a month (after taxes and other deductions). If she spends \$500 a month on rent, this is $^{500}/_{2000}$ or 0.25 (one-quarter) of her total income. Since 0.25 equals 25%, this is shown as 25% on the pie chart, and it takes up one-quarter of the total pie or circle. The size of the slice is in proportion to the percentage. Likewise, if she spends \$800 a month on food, this is $^{800}/_{2000}$ or 0.40, which is 40% of her total take-home pay. The other slices represent smaller percentages and have correspondingly smaller sizes. If we have taken into account all of her normal expenses, the sum of the percentages in the chart should add to 100%.

Why Is Scientific Notation Used?

When we need to represent very large numbers or very tiny fractions, a lot of zeros are required to locate the decimal point properly. For example, 1.2 trillion dollars (corresponding roughly to the size of our accumulated national debt several years ago) can be written as

\$1 200 000 000 000.

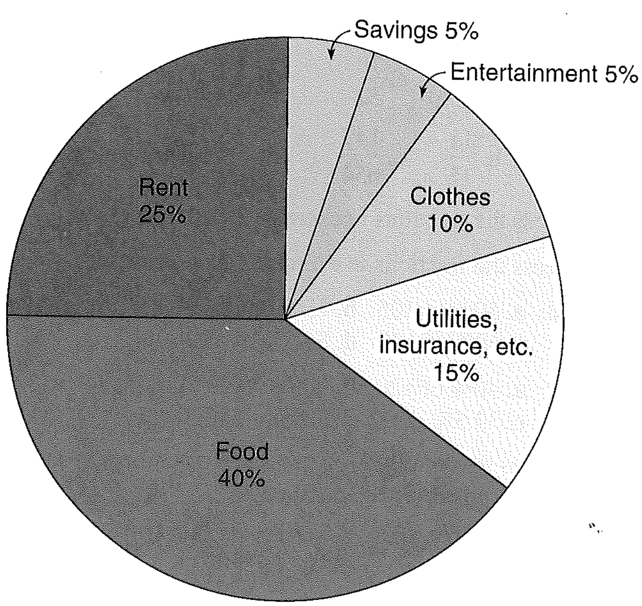

figure B.1 A pie chart showing the fractions of total take-home pay spent in different categories. The slices of the pie are in proportion to the percentages being represented.

The zeros are there only to locate the decimal point; they do not imply that all of the other numbers to the right of the 1 and 2 are exactly zero. If we count the digits to the right of the 1, we see that there are 12 (11 zeros and the digit 2).

Another way of stating this number would be to say that it is 1.2 times 1 trillion, where 1 trillion is the number 1 followed by 12 zeros. One trillion is also the number that results when you multiply 1 by ten 12 times.

$$1\,000\,000\,000\,000 = 1 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$$

The shorthand notation for a number multiplied by itself 12 times is to say that has been raised to the *power* 12, which we write as 10^{12}. The superscript represents the power to which the number has been raised, which is the number of times that you have multiplied the number by itself. We read this number as "ten to the twelfth power" or often just "ten to the twelfth."

Thus we can write the number 1.2 trillion as

1.2×10^{12}.

This notation, in which we have written the number as some number times a power of ten, is called *scientific notation.* The number 12 (the power) simply tells us how many places to the right of the indicated decimal point we would move the decimal point if we wrote out all of the zeros. Scientific notation has several advantages: it saves space, it properly indicates the accuracy or precision of the number being represented by eliminating the zeros, and it makes the number easier to manipulate in calculations involving very large or small numbers.

Some examples involving smaller numbers may help to make the concept clear. The number 586, to choose a much smaller number than 1.2 trillion, can be expressed as 5.86 times 100, or 5.86×10^2, since $10 \times 10 = 100 = 10^2$ (10 squared). The number 6 180 can be expressed as 6.18×10^3, since 10^3 (10 cubed) is 1000. The number 5 400 000 (5.4 million) can be expressed as 5.4×10^6, since 10 to the sixth power is 1 million. Several other examples are provided in table B.1. The last number listed under the positive powers of ten is the approximate mass of the Earth in kilograms.

Table B.1 also shows several decimal fractions written in scientific notation. A fraction will always have a negative exponent (negative power of ten) if the value of the fraction is less than 1. For example, the fraction 0.000 000 000 001 2 is the very tiny fraction 1.2 trillionths. It can be expressed as 1.2×10^{-12}, which is equivalent to dividing the number 1.2 by 10^{12}, or by 1 trillion. The superscript -12 tells you that you have to move the decimal point 12 places to the *left* to express the number in normal decimal form.

Taking a simpler example, the decimal fraction 0.0346 is 3.46×10^{-2} or 3.46 hundredths. Moving the decimal points two places to the left, as indicated by the power of ten, yields the original decimal fraction. The fraction 0.0079 is 7.9 thousandths or 7.9×10^{-3}. Studying the other

table B.1

Examples of Scientific Notation

Positive powers of ten		
5 460	= 5.46 times 1 thousand	$= 5.46 \times 10^3$
23 400	= 23.4 times 10 thousand	$= 2.34 \times 10^4$
6 700 000	= 6.7 times 1 million	$= 6.7 \times 10^6$
9 400 000 000	= 9.4 times 1 billion	$= 9.4 \times 10^9$
5 980 000 000 000 000 000 000 000		$= 5.98 \times 10^{24}$
Negative powers of ten (fractions)		
0.62	= 6.2 times one-tenth	$= 6.2 \times 10^{-1}$
0.0523	= 5.23 times one-hundredth	$= 5.23 \times 10^{-2}$
0.0082	= 8.2 times one-thousandth	$= 8.2 \times 10^{-3}$
0.000 0024	= 2.4 times one-millionth	$= 2.4 \times 10^{-6}$
0.000 000 0079	= 7.9 times one-billionth	$= 7.9 \times 10^{-9}$
0.000 000 000 000 000 000 16		$= 1.6 \times 10^{-19}$

examples in table B.1 should make the pattern clear. The last number in table B.1 is the value of the charge on the electron in coulombs, a quantity that arises frequently in modern physics.

The prefixes used in the metric system of units (discussed in chapter 1) are another aid to expressing very large or very small numbers. Since the prefix *mega* stands for 1 million, the quantity 1.35 Mg (megagrams) is the same as 1.35×10^6 g (10^6 is one million). Likewise, 780 nm (nanometers) is the same as 780×10^{-9} m, since the prefix *nano* means one-billionth or 10^{-9}. The values of the commonly used metric prefixes are found in table 1.3 in chapter 1. These metric prefixes and the power-of-ten scientific notation are both types of scientific shorthand used to express numbers in briefer forms.

Multiplying and Dividing Using Powers of Ten

The process of multiplying or dividing numbers written in power-of-ten notation is straightforward if you understand what they mean. It is even easier if you have a calculator that handles scientific notation—you just punch the numbers in and push the appropriate function key. Some understanding of their meaning, though, can be useful for checking your results.

Suppose, for example, that we multiply the number 3.4×10^3 (3400) by 100 (10^2). Multiplying by 100 adds two zeros to the original number, yielding 340 000, as you can quickly check by doing this operation on a calculator or by direct multiplication. Thus

$$(3.4 \times 10^3) \times (10^2) = 3.4 \times 10^5.$$

In other words, the powers of ten add (3 + 2 = 5). If we divided by 100, we would remove two zeros:

$$\frac{3.4 \times 10^3}{10^2} = 3.4 \times 10^1 = 34.$$

In this case, the exponent of the denominator is subtracted from the exponent of the number being divided (3 − 2 = 1).

The rules for these operations are thus

1. When numbers are multiplied, the powers of ten add.
2. When numbers are divided, the power of the denominator is subtracted from the power of the numerator.

These rules are valid regardless of whether the powers are positive or negative. Thus

$$(3 \times 10^6) \times (2 \times 10^{-4}) = 6 \times 10^2 = 600$$

since 6 + (−4) = 2. This should make sense to you since multiplying by a fraction (a number with a negative power of ten) results in a smaller number than the number being multiplied.

Exercises

If any of these ideas are unfamiliar—or even if they are familiar but you are rusty in using them—working some or all of these exercises will help to build your confidence. The answers to the odd-numbered exercises are found in appendix D.

(Exercises 1 through 4) Express these numbers as decimal fractions:

1. **a.** $\frac{6}{10}$ **b.** $\frac{52}{100}$ **c.** $\frac{874}{1000}$ **d.** $\frac{5}{10\,000}$
2. **a.** $\frac{72}{100}$ **b.** $\frac{7}{10}$ **c.** $\frac{83}{10\,000}$ **d.** $\frac{45}{1000}$
3. **a.** $\frac{1}{4}$ **b.** $\frac{5}{8}$ **c.** $\frac{16}{52}$ **d.** $\frac{312}{914}$ (Use a calculator.)
4. **a.** $\frac{3}{7}$ **b.** $\frac{11}{15}$ **c.** $\frac{147}{654}$ **d.** $\frac{65}{150}$ (Use a calculator.)
5. Express the fractions in exercise 3 as percentages.
6. Express the fractions in exercise 4 as percentages.
7. Find: **a.** 50% of 105 **b.** 75% of 48 **c.** 60% of 180 **d.** 85.2% of 100
8. Find: **a.** 40% of 120 **b.** 90% of 400 **c.** 33.3% of 90 **d.** 70% of 540

(Exercises 9 through 12) Express these numbers in scientific notation (power-of-ten notation):

9. **a.** 5475 **b.** 200 000 **c.** 67 000 **d.** 35 000 000 000
10. **a.** 3560 **b.** 78 500 **c.** 622 000 **d.** 9 100 000
11. **a.** 0.0065 **b.** 0.000 333 **c.** 0.000 001 5 **d.** 0.000 000 065
12. **a.** 0.075 **b.** 0.000 45 **c.** 0.000 003 2 **d.** 0.000 89