PHYC 521: Graduate Quantum Mechanics I

Fall 2009

Homework Assignment #6

(Due October 28)

1- Exercise 16.1.3, Shankar, 2nd edition, page 434.

2- Consider a particle on a line (we choose that to be the x axis). We want to prove that there is no degeneracy in the ground state regardless of the potential $V(x)$.

(a) First prove that the ground state wavefunction $\psi_0(x)$, with energy E_0, has the same sign for all x. To do this, assume that $\psi_0(x)$ changes sign at the point x_0 (so that it has a node at x_0). Define a new wavefunction $\psi_0'(x) = |\psi_0(x)|$, and show that $\langle\psi_0'|H|\psi_0'\rangle = \langle\psi_0|H|\psi_0\rangle = E_0$. (Hint: note that $\langle\psi_0'|P^2|\psi_0'\rangle = \langle P\psi_0'|P\psi_0'\rangle$; P is the momentum operator.)

(b) Show that $|\psi_0'\rangle$ does not obey the equation $H|\psi_0'\rangle = E_0|\psi_0'\rangle$. Therefore we must have $\langle\psi_0'|H|\psi_0'\rangle > E_0$. This contradicts with (a), which implies that $\psi_0(x)$ cannot change sign.

(c) Now assume that there are two orthogonal eigenstates $|\psi_0\rangle$ and $|\psi_1\rangle$ that have the same eigenvlaue E_0. Then we know from parts (a),(b) that neither $\psi_0(x)$ nor $\psi_1(x)$ can change sign. Use this to show that $\langle\psi_0|\psi_1\rangle \neq 0$. Thus the ground state is unique.

3- Consider two trial functions

$$\begin{aligned}
\psi_0(x) &= A(x+a) \quad & -a \le x < 0 \\
\psi_0(x) &= A(a-x) \quad & 0 < x \le a
\end{aligned}$$

and

$$\begin{aligned}
\psi_1(x) &= B(x+a) \quad & -a \le x < \frac{-a}{2} \\
\psi_1(x) &= -Bx \quad & \frac{-a}{2} < x < \frac{a}{2} \\
\psi_1(x) &= B(x-a) \quad & \frac{a}{2} < x \le a\,,
\end{aligned}$$

both of which vanish for $x < -a$ and $x > a$.

(a) Determine A and B such that $\psi_0(x)$ and $\psi_1(x)$ are normalized wavefunctions.

(b) Find $\langle\psi_0|H|\psi_0\rangle$ and $\langle\psi_1|H|\psi_1\rangle$, and compatre them with the energy of the ground state E_0 and the first excited state E_1 of a free particle inside the box $-a < x < a$. Which of ψ_0 and ψ_1 leads to a better approximation of E_0? Which one gives a better approximation of E_1? Comment.

(c) Choose a trial function $\psi_2(x)$ along this line to estimate the energy of the second excited state E_2. State your reason. (No claculation needed, just sketch the trial function that you choose).