PHYC 511
spring 2019

(1)

01/23/2019

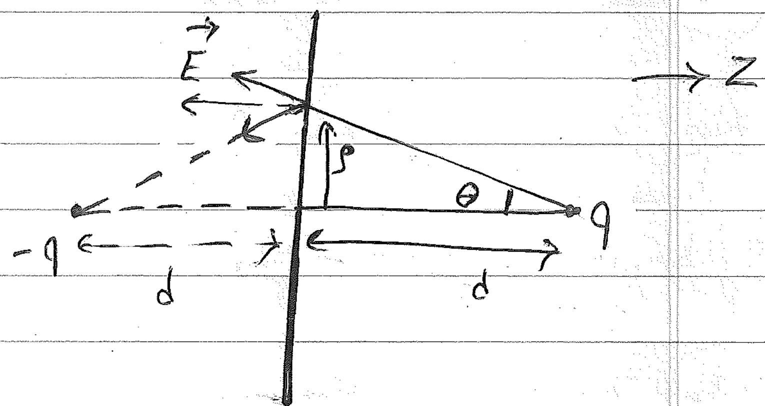
Problem Session 2

(1) Problem 2.1, Jackson, parts (a) - (e).

(2) Problem 2.2, Jackson.

(1)

(a) Being a conductor, the surface charge density at a distance ρ is:



$$E_n(\rho) = \frac{\sigma}{\varepsilon_0}$$

$$E_n(\rho) = \frac{-2q}{4\pi\varepsilon_0} \frac{1}{\rho^2+d^2} \cos\theta = \frac{-qd}{2\pi\varepsilon_0 (\rho^2+d^2)^{3/2}} \Rightarrow \boxed{\sigma = -\frac{qd}{2\pi(\rho^2+d^2)^{3/2}}}$$

(b) The force on the plate is equal in magnitude and opposite in direction to the force on charge q. This force is easily obtained by Coulomb's law between the charge q and the image charge $-q$:

$$\vec{F} = \frac{-q^2}{4\pi\varepsilon_0(2d)^2}\hat{z} = \boxed{\frac{-q^2}{16\pi\varepsilon_0 d^2}\hat{z}}$$

(c) In order to find the force on the plane from charge q, we should take into account the electric field on the plane that is due to q only. Hence:

$$\vec{F}_{tot} = \int_0^\infty \frac{1}{2}\sigma E_n(\rho)\, 2\pi\rho\, d\rho\, \hat{z} = \int_0^\infty \frac{2\pi\, q^2 d^2}{8\pi^2\varepsilon_0(\rho^2+d^2)^3}\rho\, d\rho\, \hat{z} = \int_0^\infty \frac{q^2 d^2}{4\pi\varepsilon_0(\rho^2+d^2)^3}\rho\, d\rho\, \hat{z}$$

$u \equiv \rho^2 + d^2 \Rightarrow du \propto \rho\, d\rho$

Thus:

$$\vec{F}_{tot} = \int_{d^2}^\infty \frac{q^2 d^2\, du}{8\pi\varepsilon_0 u^3}\hat{z} = \boxed{\frac{q^2}{16\pi\varepsilon_0 d^2}\hat{z}}$$

This is in agreement with the answer to part (b).

(d) The work done is:

$$W = \int_d^\infty \vec{F}\, dz = \int_d^\infty \frac{q^2}{16\pi\varepsilon_0 z^2} dz = \boxed{\frac{q^2}{16\pi\varepsilon_0 d}}$$

(e) The potential energy between q and -q simply is:

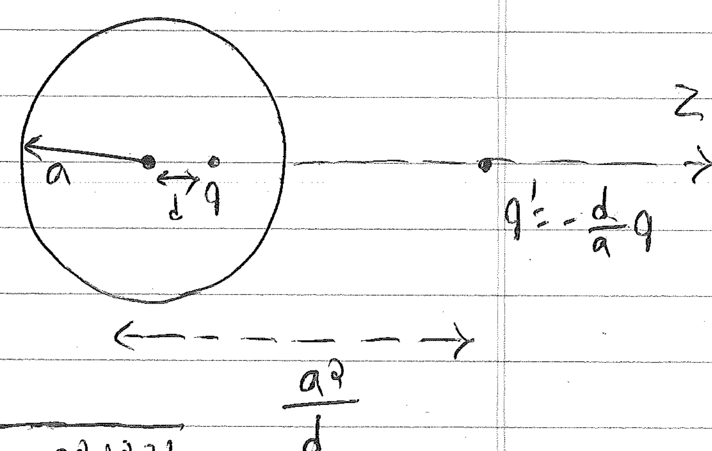
$$\boxed{U = \frac{-q^2}{4\pi\varepsilon_0(2d)} = \frac{-q^2}{8\pi\varepsilon_0 d}}$$

In general, one expects that $U = -W$. The discrepancy between

U in part (c) and W in part (d) is due to the fact that W is the work done on q only, while U is the total potential energy in the system. It therefore includes the work done on the charges on plate whose distribution changes as q is moving away from the plate.

(2)

(a) At a point (x, y, z) inside the sphere, we have:



$$\Phi = \frac{q}{4\pi\varepsilon_0 \left[x^2 + y^2 + (z-d)^2\right]^{1/2}} - \frac{d\,q}{4\pi\varepsilon_0 a \left[x^2 + y^2 + \left(z - \frac{a^2}{d}\right)^2\right]^{1/2}}$$

$$\Rightarrow \Phi = \frac{q}{4\pi\varepsilon_0 \left[r^2 + d^2 - 2r\cos\theta\, d\right]^{1/2}} - \frac{d\,q}{4\pi\varepsilon_0 a \left[r^2 + \frac{a^4}{d^2} - 2r\cos\theta \frac{a^2}{d}\right]^{1/2}}$$

(b) Being a conductor, the surface charge density induced on the sphere is:

$$\sigma = \varepsilon_0 E_r(r=a) = -\varepsilon_0 \frac{\partial \Phi}{\partial r}\Big|_{r=a}$$

Hence:

$$\boxed{\sigma = \frac{q(a-d\cos\theta)}{4\pi\,[a^2+d^2-2ad\cos\theta]^{3/2}} - \frac{dq\left(a-\frac{a^2}{d}\cos\theta\right)}{4\pi\left[a^2+\frac{a^4}{d^2}-2\frac{a^3}{d}\cos\theta\right]^{3/2}}}$$

(c) The magnitude and direction of the force on q follows from the Coulomb's law:

$$\vec{F} = \frac{q \times \frac{d}{a} q}{4\pi\varepsilon_0 \left(\frac{a^2}{d}-d\right)^2}\,\hat{z} = \boxed{\frac{q^2 d^3}{4\pi\varepsilon_0 a (a^2-d^2)^2}\,\hat{z}}$$

(d) If the sphere has potential V, the result in part (a) will change by a value V. The results in part (b) and (c) will not change however because σ and $\vec{F}$ depend on the derivatives of the potential.

If the sphere has a total charge Q and is isolated, then a charge $Q+q$ will be accumulated uniformly on its outer surface. The reason being that a total charge of $-q$ will be accumulated on the inner surface in order to have $\vec{E} = 0$ in the conductor. This charges the potential inside the sphere by a value $\frac{Q+q}{4\pi\varepsilon_0 b}$, where b is the radius of the outer surface of the sphere. The results in parts (b) and (c) will remain unchanged however.