

parts

(1) Problem 7.15, <sup>parts</sup>(a) and (b), Jackson.

(2) Problem 7.20, Jackson.

(2)

(1) (a) The transit time of a pulse over distance  $dz$  is:

$$dt(\omega) = \frac{dz}{v_g(\omega)} \quad , \quad v_g(\omega) = \frac{d\omega}{dk}$$

Thus:

$$dt(\omega) = \frac{dk}{d\omega} dz = dz \frac{1}{c} \frac{d}{d\omega} \left[ \omega \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \right]$$

"  $1 - \frac{\omega_p^2}{2\omega^2} + O\left(\frac{\omega_p}{\omega}\right)^4$

Then:

$$dt(\omega) \approx \frac{dz}{c} \left( 1 + \frac{\omega_p^2}{2\omega^2} \right) = \frac{dz}{2c\omega^2} \frac{\hbar e(z) e^2}{\epsilon_0 m_e} + \frac{dz}{c} \Rightarrow t(\omega) = \int_{z=0}^R dt(\omega) \Rightarrow$$

$$t(\omega) \approx \frac{R}{c} + \frac{1}{2\epsilon_0 c \omega^2 m_e} \int_0^R \hbar e(z) dz$$

(b) We have:

$$\delta\theta = \frac{1}{2} |\phi_+ - \phi_-|$$

Where  $\phi_{\pm}$  denote phase pick up by  $\pm$  helicity states. For propagation through distance  $dz$ :

$$\phi_{\pm} = \frac{\omega}{c} \sqrt{\frac{\epsilon_{\pm}}{\epsilon_0}} dz \approx dz \frac{\omega}{c} \sqrt{1 - \frac{\omega_p^2}{\omega^2} \mp \frac{\omega_B}{\omega} \frac{\omega_p^2}{\omega^2} \hat{b} \cdot \hat{n}} = dz \frac{\omega}{c} \left[ -\frac{\omega_p^2}{\omega^2} \mp \frac{1}{2} \frac{\omega_p^2}{\omega^2} \frac{e\vec{B} \cdot \hat{n}}{m_e \omega B_{||}} + O\left(\frac{\omega_p}{\omega}\right)^4 \right]$$

Therefore:

$$\delta\theta(z \rightarrow z+dz) \approx \frac{1}{2} dz \frac{\omega}{c} \frac{\omega p^d}{\omega^2} \frac{e B_{||}(z)}{\omega m_e} = \frac{e^3}{2c\omega^3 \epsilon_0 m_e^2} n_e(z) B_{||}(z) dz$$

The full linear polarization rotation is obtained to be;

$$\delta\theta(z_0 \rightarrow R) = \frac{e^3}{2\epsilon_0 c \omega^3 m_e^2} \int_{z_0}^R n_e(z) B_{||}(z) dz$$

(2) (a) Each frequency  $\omega$  is associated with its propagation constant

$k(\omega) = \frac{\omega}{c} n(\omega)$ . In one dimension, a plane wave of frequency  $\omega$

can propagate along  $+x$  or  $-x$  direction. Hence, a plane wave has

the following general form:

$$A(\omega) e^{i(k(\omega)x - \omega t]} + B(\omega) e^{i[k(\omega)x + \omega t]}$$

Superposing monochromatic plane waves then generates a general

wave as follows:

$$U(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega [A(\omega) e^{i\frac{\omega}{c} n(\omega)x} + B(\omega) e^{-i\frac{\omega}{c} n(\omega)x}] e^{-i\omega t}$$

(b) If  $U(x,t)$  is real, we will have:

$$U^*(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega [A^*(\omega) e^{-i\frac{\omega}{c} n^*(\omega)x} + B^*(\omega) e^{i\frac{\omega}{c} n^*(\omega)x}] e^{i\omega t}$$

using  $\omega = -\omega'$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega' [A^*(-\omega') e^{i\frac{\omega'}{c} n^*(-\omega')x} + B^*(-\omega') e^{-i\frac{\omega'}{c} n^*(-\omega')x}] e^{-i\omega' t} =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega [A^*(-\omega) e^{i\frac{\omega}{c} n^*(-\omega)x} + B^*(-\omega) e^{-i\frac{\omega}{c} n^*(-\omega)x}] e^{-i\omega t}$$

Comparing with  $u(\omega, t)$  in part (a), we get:

$$A(\omega) = A^*(-\omega), \quad B(\omega) = B^*(-\omega), \quad h(\omega) = h^*(-\omega) \Rightarrow h(-\omega) = h^*(\omega)$$

(c) From part (a), we have:

$$u(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega [A(\omega) + B(\omega)] e^{-i\omega t}$$

$$\frac{\partial u}{\partial \eta}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} i\frac{\omega}{c} h(\omega) [A(\omega) - B(\omega)] e^{-i\omega t}$$

By taking the inverse Fourier transform of these two relations and solving for  $A(\omega)$  and  $B(\omega)$ , we find:

$$A(\omega) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt e^{i\omega t} \left[ u(\omega, t) - \frac{ic}{\omega h(\omega)} \frac{\partial u}{\partial \eta}(\omega, t) \right]$$

$$B(\omega) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt e^{i\omega t} \left[ u(\omega, t) + \frac{ic}{\omega h(\omega)} \frac{\partial u}{\partial \eta}(\omega, t) \right]$$