Lecture 5 — 01/30/2019

Review of Electrostatics (Cont'd)

Spherical Coordinates

$$\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\Phi(r,\theta,\phi) = R(r)\, P(\theta)\, Q(\phi) \Rightarrow \frac{1}{R(r)}\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{r^2 \sin\theta}\frac{1}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{1}{Q}\frac{d^2 Q}{d\phi^2} = 0 \Rightarrow \underbrace{\frac{1}{R(r)\, r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)}_{\text{function of } r \text{ only}} +$$

$$\frac{1}{r^2}\Bigg[\underbrace{\frac{1}{\sin\theta P}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \frac{1}{\sin^2\theta\, Q}\frac{d^2 Q}{d\phi^2}}_{\text{function of } \theta, \phi \text{ only}}\Bigg] = 0$$

Each of these functions must be a constant:

$$\begin{cases} \dfrac{1}{Q}\dfrac{d^2 Q}{d\phi^2} = -m^2 \Rightarrow Q(\phi) \propto e^{\pm i m \phi} \\[2ex] \dfrac{1}{\sin\theta P}\dfrac{d}{d\theta}\left(\sin\theta \dfrac{dP}{d\theta}\right) - \dfrac{m^2}{\sin^2\theta} + l(l+1) = 0 \\[2ex] \dfrac{1}{r^2 R}\dfrac{d}{dr}\left(r^2 \dfrac{dR}{dr}\right) - \dfrac{l(l+1)}{r^2} = 0 \Rightarrow R \propto r^l,\ r^{-(l+1)} \end{cases}$$

Solutions to the second equation are associated Legendre functions.

$P_\ell^m(\cos\theta)$, $Q_\ell^m(\cos\theta)$. However, considering the entire range of the polar angle $0 \le \theta \le \pi$, these functions are singular at $\theta = 0, \pi$ unless $\ell = 0, 1, 2, \ldots$ and $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$. Even then only $P_\ell^m(\cos\theta)$ is finite, which is called the associated Legendre polynomial:

$$P_\ell^m(\cos\theta) = (-1)^m \sin^{|m|}\theta \frac{d^{|m|}}{d\cos\theta^{|m|}} P_\ell(\cos\theta)$$

$P_\ell(x)$ are the Legendre polynomials that satisfy the following differential equation:

$$(1-x^2)\frac{d^2 P_\ell}{dx^2} - 2x\frac{dP_\ell}{dx} + \ell(\ell+1)P_\ell = 0$$

They are given by the Rodriguez formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

The orthogonality relation for P_ℓ is:

$$\int_{-1}^{+1} P_\ell(x) P_{\ell'}(x)\,dx = \frac{2\delta_{\ell\ell'}}{2\ell+1}$$

We can write $P_\ell^m(\cos\theta)e^{\pm im\phi}$ in terms of the "spherical harmonics":

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi}$$

Some of the useful properties of $Y_{\ell m}$ are as follows:

$$Y_{\ell m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi)$$

$$\iint Y^*_{\ell m}(\theta,\phi)\, Y_{\ell' m'}(\theta,\phi)\sin\theta\, d\theta\, d\phi = \delta_{\ell\ell'}\,\delta_{mm'} \qquad \text{(orthogonality)}$$

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} Y^*_{\ell m}(\theta,\phi)\, Y_{\ell m}(\theta',\phi') = \frac{1}{\sin\theta}\,\delta(\theta-\theta')\,\delta(\phi-\phi')$$

We can also write $\dfrac{1}{|\vec{x}-\vec{x}'|}$ in terms of the spherical harmonics:

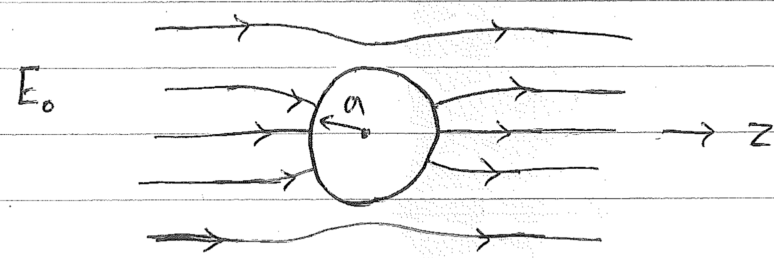
$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}}\cdot\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{+\ell} Y^*_{\ell m}(\theta,\phi)\, Y_{\ell m}(\theta',\phi')$$

Here $r_<$ is the smaller of $|\vec{x}|, |\vec{x}'|$, while $r_>$ is the larger of the two.

The general solution for Φ can then be written as:

$$\Phi(r,\theta,\phi) = \sum_{\ell,m}\left[A_{\ell m} r^{\ell} + B_{\ell m} r^{-(\ell+1)}\right] Y_{\ell m}(\theta,\phi)$$

Example: Conducting sphere in a uniform electric field.



Choosing the z axis as shown above, we can exploit the azimuthal symmetry, which implies no ϕ dependence. This simplifies the situation as only terms with $m=0$ are now present in Φ:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) Y_{lm}(\theta,\phi)$$

Asymptotic behavior of the electric field implies that $A_l = 0$ for $l \geq 2$. Also, since the sphere is isolated and has no net charge, we have $B_0 = 0$. Having an equipotential surface at $r=a$ requires that:

$$A_1 a + B_1 a^{-2} = 0 \Rightarrow B_1 = A_1 a^3$$

Note that $\Phi \to -E_0 z$ as $|z| \to \infty$. Considering that $z = r\cos\theta$, we then find:

$$\boxed{\Phi(r,\theta) = -E_0 r\cos\theta + E_0 \frac{a^3}{r^2}\cos\theta} \quad *$$

This is the superposition of the potential due to a uniform electric field and that due to an electric dipole. This can be seen by calculating the induced surface charge density on the sphere:

$$\sigma(\theta) = \epsilon_0 E_n(\theta) = -\epsilon_0 \frac{\partial \Phi}{\partial r}\Big|_{r=a} = 3\epsilon_0 E_0 \cos\theta$$

The dipole moment of the sphere thus follows:

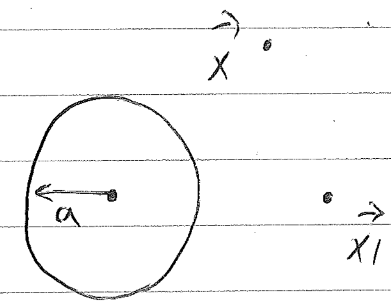
$$\vec{P} = a^3 \int_0^{2\pi}\int_0^{\pi} \sigma(\theta) \cos\theta \sin\theta \, d\theta \, d\phi \, \hat{z} = 4\pi \epsilon_0 E_0 a^3 \hat{z}$$

The electric potential due to such a dipole is $\frac{\vec{P} \cdot \vec{x}}{r^2}$, which gives exactly the same result as in the second term on the right-hand side of * in above.

We now consider applications of the spherical harmonics in computing Green's functions for problems with spherical boundaries.

Green's Functions for the Exterior Problem

As we saw before:



$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + \tilde{G}(\vec{x}, \vec{x}')$$

Where:

$$\nabla^2 \tilde{G}(\vec{x}, \vec{x}') = 0$$

The general solution for $\tilde{G}$ is given by:

$$\tilde{G}(\vec{x}, \vec{x}') = \sum_{l,m} B_{l,m} r^{-(l+1)} Y_{lm}(\theta, \phi)$$

We note that r^{ℓ} terms vanish due to local nature of the boundary as $\tilde{G}$ is the potential due to the induced charges on the boundary.

Hence:

$$G(\vec{x},\vec{x}') = \sum_{l,m} B_{l,m} r^{-(l+1)} Y_{lm}(\theta,\phi) + \frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m} B_{l,m} r^{-(l+1)} Y_{lm}(\theta,\phi)$$

$$+ \sum_{l,m} \frac{r_<^{\ell}}{r_>^{\ell+1}} \cdot \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi) Y^*_{lm}(\theta',\phi') = \sum_{l,m} \left[\frac{4\pi}{2l+1} \frac{r_<^{\ell}}{r_>^{l+1}} Y^*_{lm}(\theta',\phi') \right.$$

$$\left. + B_{l,m} \frac{1}{r^{l+1}} \right] Y_{lm}(\theta,\phi)$$

For the Dirichlet problem, we must have $G_D(\vec{x},\vec{x}')\big|_{r=a} = 0$.

This requires that:

$$\frac{4\pi}{2l+1} \frac{a^{\ell}}{r'^{l+1}} Y^*_{lm}(\theta',\phi') + B_{l,m} \frac{1}{a^{l+1}} = 0 \quad (r_< = a,\ r_> = r')$$

Therefore:

$$\boxed{B_{l,m} = \frac{-4\pi}{2l+1} \frac{a^{2l+1}}{r'^{l+1}} Y^*_{lm}(\theta',\phi')}$$

For the Neumann problem, we must have $\dfrac{\partial G_N(\vec{x},\vec{x}')}{\partial r}\Big|_{r=a} = 0$ due to boundary at infinity.

This condition determines the coefficients B_{ℓ} in this case.

Green's Functions for the Interior Problem

In this case, the terms $A_\ell r^\ell$ are present in the expression for $\tilde{G}(\vec{x}, \vec{x}')$. The Dirichlet problem is similar to that for the exterior problem as the condition $G_D(\vec{x}, \vec{x}')\big|_{r=a} = 0$ determines the coefficients A_ℓ. However, the Neumann problem is more involved in this case. This is because $\frac{\partial G_N(\vec{x}, \vec{x}')}{\partial r}\Big|_{r=a} = \frac{-4\pi}{4\pi a^2} = -\frac{1}{a^2}$. This affects the $\ell = m = 0$ contribution, which is a constant. For $\ell \neq 0$, the coefficients A_ℓ are determined analogously to the exterior problem.