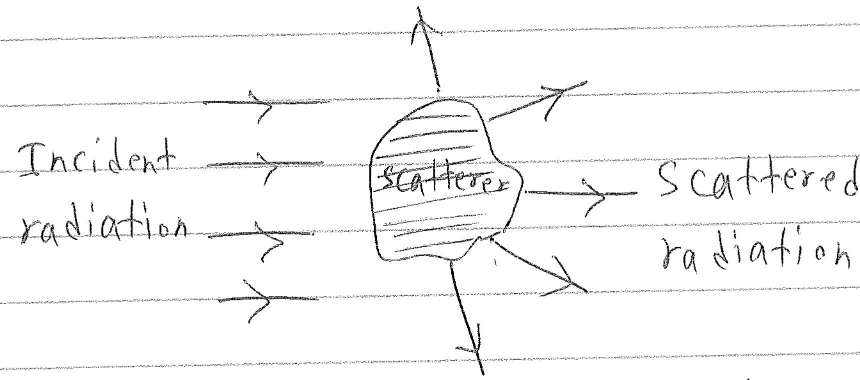
Lecture 22 04/17/2019

Scattering

In the general scattering problem, a scatterer interacts with incident radiation and radiates a spherical wave as a result of the interaction.



For monochromatic radiation at frequency ω, we have:

$$\vec{A}(\vec{x}) = \vec{A}_{inc}(\vec{x}) + \frac{\mu_0}{4\pi}\int \vec{J}^{tot}_{ind}(\vec{x}')\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}d^3x'$$

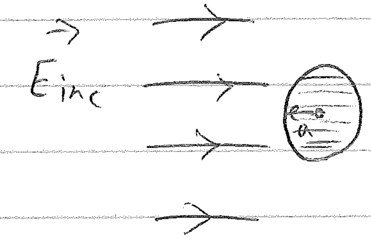
Here, $\vec{J}^{tot}_{ind}(\vec{x}')e^{-i\omega t}$ is the total induced current resulting from the interaction. It includes both bound and macroscopic currents $\frac{\partial \vec{P}_{ind}}{\partial t}$, $\vec{\nabla}\times\vec{M}_{ind}$, and $\vec{J}_{ind}$.

In the long wavelength limit, when the scatterer is small in size compared to the wavelength, one may use the quasistatic approximation of the fields where those fields given

given by electrostatics and magnetostatics are multiplied by an $e^{-i\omega t}$ factor (in the complex notation).

Example: Scattering by a small dielectric sphere.

$a << \lambda$



$\varepsilon = \varepsilon_0 \, \varepsilon_r$

In this case, the sphere develops a uniform polarization density given by:

$$\vec{\rho} = 3\varepsilon_0 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) \vec{E}_{inc} \; e^{-i\omega t}$$

It therefore acts as a dipole with dipole moment:

$$\vec{P} = \frac{4}{3}\pi a_3 \vec{\rho} = 4\pi\varepsilon_0 \, a^3 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\right) \vec{E}_{inc}$$

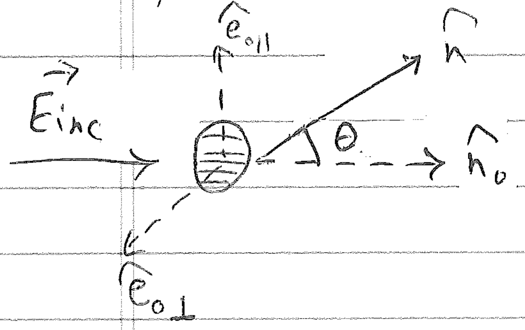
In the far zone, the scattered radiation takes the following form:

$$\vec{E}_{sc} = \frac{k^2}{4\pi\varepsilon_0} (\hat{n} \times \vec{P}) \times \hat{n} \; \frac{e^{ikr}}{r}$$

The total $\vec{E}$ field thus follows:

$$\vec{E} = \vec{E}_{inc} \, e^{i(\vec{k}_0 \cdot \vec{r} - \omega t)} + k^2 a^3 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) \frac{e^{i(kr - \omega t)}}{r} (\hat{n} \times \vec{E}_{inc}) \times \hat{n}$$

The angle between the scattered and incident directions, denoted by θ, is called the "scattering angle", and the plane formed by the two directions is called the "plane of scattering".



Because the incident and scattered fields each have two possible polarizations, the scattering is described in terms of four variables, $\hat{n}_0, \hat{e}_0; \hat{n}, \hat{e}$, where $\hat{e}_0$ and $\hat{e}$ are the polarization vectors defined relative to the plane of scattering.

Differential scattering cross section is defined as:

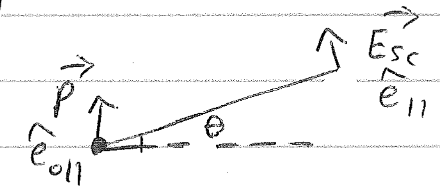
$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{e}; \hat{n}_0, \hat{e}_0) = \frac{\text{Power scattered per unit solid angle in polarization } \hat{e}}{\text{Incident intensity for polarization along } \hat{e}_0}$$

Thus:

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{e}; \hat{n}_0, \hat{e}_0) = \frac{r^2 \, |\hat{e}^* \cdot \vec{E}_{sc}|^2}{|\hat{e}_0^* \cdot \vec{E}_{inc}|^2}$$

For $\hat{e}_0$ linearly polarized in the plane of scattering, we have:

$$\frac{d\sigma}{d\Omega}(\hat{n},\hat{e}_{\parallel};\hat{n}_0,\hat{e}_{0\parallel}) = r^2\,\frac{|\hat{e}_{\parallel}^{*}\cdot\vec{E}_{sc}|^2}{|\vec{E}_{inc}|^2} = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2\left|[(\hat{n}\times\hat{e}_{0\parallel})\times\hat{n}]\cdot\hat{e}_{\parallel}^{*}\right|^2$$



$$= k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2\left|(\hat{n}\times\hat{e}_{0\parallel})\cdot(\hat{n}\times\hat{e}_{\parallel}^{*})\right|^2$$

But:

$$(\hat{n}\times\hat{e}_{0\parallel})\cdot(\hat{n}\times\hat{e}_{\parallel}^{*}) = (\hat{n}\cdot\hat{n})(\hat{e}_{0\parallel}\cdot\hat{e}_{\parallel}^{*}) - (\hat{n}\cdot\hat{e}_{0\parallel})(\overset{0}{\cancel{\hat{n}\cdot\hat{e}_{\parallel}^{*}}}) = \cos^2\theta$$

This results in:

$$\frac{d\sigma}{d\Omega}(\hat{n},\hat{e}_{\parallel};\hat{n}_0,\hat{e}_{0\parallel}) = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2\cos^2\theta$$

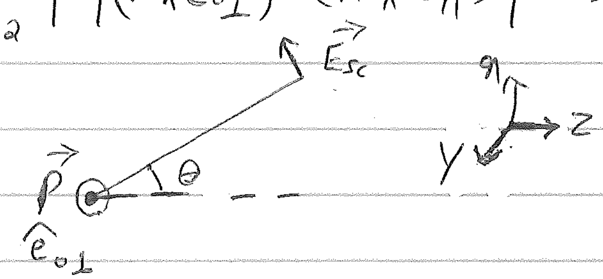
Similarly, since $\vec{E}_{sc}$ is fully polarized along $\hat{e}_{\parallel}$, we find:

$$\frac{d\sigma}{d\Omega}(\hat{n},\hat{e}_{\perp};\hat{n}_0,\hat{e}_{0\parallel}) = 0$$

For $\hat{e}_0$ linearly polarized normal to the plane of scattering, we have:

$$\frac{d\sigma}{d\Omega}(\hat{n},\hat{e}_{\parallel};\hat{n}_0,\hat{e}_{0\perp}) = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2\left|(\hat{n}\times\hat{e}_{0\perp})\cdot(\hat{n}\times\hat{e}_{\parallel}^{*})\right|^2 = 0$$

$\hat{n} = \sin\theta\,\hat{x} + \cos\theta\,\hat{z}$



$\hat{e}_{0\perp} = \hat{y}$, $\hat{e}_{\parallel} = \cos\theta\,\hat{x} - \sin\theta\,\hat{z}$

This is simply because $\vec{E}_{sc}$ is perpendicular to the plane of scattering.

And:

$$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{e}_\perp; \hat{n}_0, \hat{e}_{0\perp}) = k^4 a^6 \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2$$

For unpolarized incident radiation, the scattering cross section is defined as an average over the cross sections for the two independent incident polarizations. Then:

$$\frac{d\sigma_{\parallel}}{d\Omega} = \frac{1}{2}\left[\frac{d\sigma}{d\Omega}(\hat{n}, \hat{e}_{\parallel}; \hat{n}_0, \hat{e}_{0\parallel}) + \frac{d\sigma}{d\Omega}(\hat{n}, \hat{e}_{\parallel}; \hat{n}_0, \hat{e}_{0\perp})\right] \Rightarrow$$

$$\frac{d\sigma_{\parallel}}{d\Omega} = \frac{1}{2} k^4 a^6 \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 \cos^2\theta$$

Similarly:

$$\frac{d\sigma_\perp}{d\Omega} = \frac{1}{2}\left[\frac{d\sigma}{d\Omega}(\hat{n}, \hat{e}_\perp; \hat{n}_0, \hat{e}_{0\parallel}) + \frac{d\sigma}{d\Omega}(\hat{n}, \hat{e}_\perp; \hat{n}_0, \hat{e}_{0\perp})\right] \Rightarrow$$

$$\frac{d\sigma_\perp}{d\Omega} = \frac{1}{2} k^4 a^6 \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 \left(\neq \frac{d\sigma_{\parallel}}{d\Omega}\right)$$

The fact that $\frac{d\sigma_\perp}{d\Omega} \neq \frac{d\sigma_{\parallel}}{d\Omega}$ implies that the scattered radiation has a net polarization even when the incident field is unpolarized.

This is the process of "repolarization". The degree of repolarization is represented by the contrast ration:

$$\Pi = \frac{\left| \frac{d\sigma_\perp}{d\Omega} - \frac{d\sigma_\parallel}{d\Omega} \right|}{\left| \frac{d\sigma_\perp}{d\Omega} + \frac{d\sigma_\parallel}{d\Omega} \right|} = \frac{1-\cos^2\theta}{1+\cos^2\theta}$$

The differential cross section when both polarizations are included is:

$$\frac{d\sigma_{tot}}{d\Omega} = \frac{d\sigma_\perp}{d\Omega} + \frac{d\sigma_\parallel}{d\Omega} = \frac{1}{2} k^4 a^6 \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 (1+\cos^2\theta)$$

The total scattering cross section is given by:

$$\sigma_{tot} = \int \frac{d\sigma_{tot}}{d\Omega}\, d\Omega = \frac{1}{2} k^4 a^6 \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 \int (1+\cos^2\theta)\, d\Omega \Rightarrow$$

$$\sigma_{tot} = \frac{8\pi}{3} k^4 a^6 \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2$$