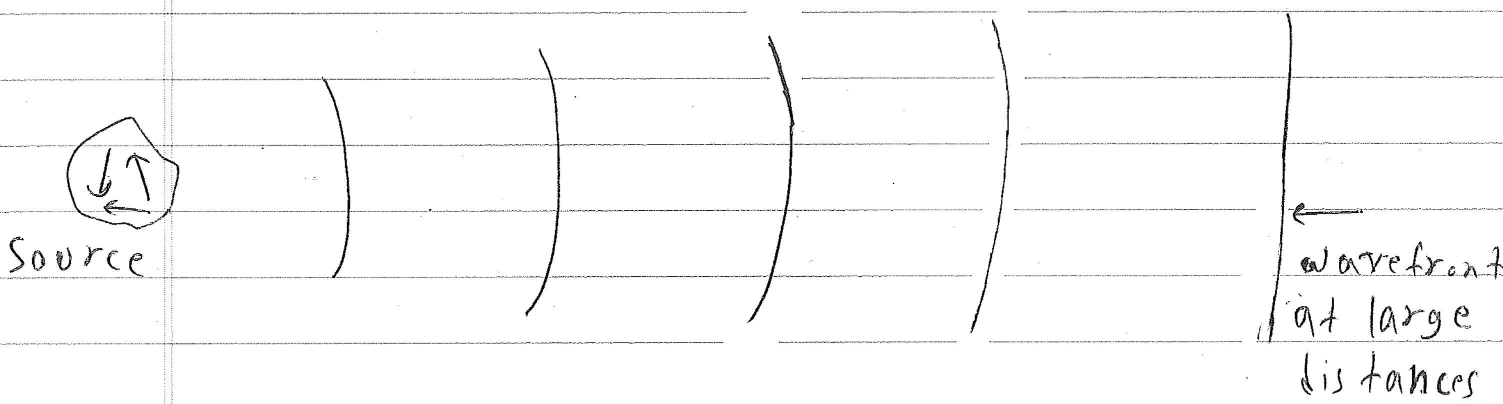
Lecture 15 03/18/2019

Electromagnetic Plane Waves

For very far away sources of radiation, the wavefronts (i.e., surfaces of constant phase and amplitude) are approximately planes:



Moreover, a general wave can be constructed from superposition of plane waves as a Fourier integral. It is therefore important to discuss the plane wave solutions to Maxwell equations in source-free space.

The $\vec{E}$ and $\vec{B}$ fields in homogeneous linear media, which contain no free charges or currents, obey the following equations:

$$\left(\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2}\right)\vec{E} = 0 \quad , \quad \left(\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2}\right)\vec{B} = 0$$

$$\vec{\nabla}\cdot\vec{E} = 0 \quad , \quad \vec{\nabla}\cdot\vec{B} = 0$$

Considering harmonic time dependence, and using complex notation, we can write solutions to the wave equations as:

$$\vec{E}(\vec{x},t) = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad , \quad \vec{B}(\vec{x},t) = \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

Where:

$$k^2 \equiv \vec{k}\cdot\vec{k} = \mu\epsilon\,\omega^2$$

For $\mu\epsilon$ real and positive, we can define $\sqrt{\mu\epsilon} = \frac{n}{c}$ with $\frac{c}{n}$ the speed of wave propagation in the medium ($n \geq 1$).

We note that:

$$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E}_0 = 0 \Rightarrow \boxed{\vec{E} \perp \vec{k}}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{k}\cdot\vec{B}_0 = 0 \Rightarrow \boxed{\vec{B} \perp \vec{k}}$$

Also:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{k}\times\vec{E} = \omega\vec{B} \Rightarrow \boxed{\vec{E} \perp \vec{B} \, , \; |\vec{E}| = \frac{c}{n}|\vec{B}|}$$

This implies that the electromagnetic plane wave is a <u>transverse</u> wave for which $\vec{E}$ and $\vec{B}$ are also perpendicular to each other.

An important point to note is that real $\mu\varepsilon$ (hence real n) does not imply that $\vec{k}$ is necessarily real. To see this, we can write:

$$\vec{k} = \frac{\omega}{c} n \hat{s} \qquad (\hat{s}: \text{a unit vector})$$

$$\vec{k}\cdot\vec{k} = \frac{\omega^2 n^2}{c^2} \Rightarrow (\hat{s}_R + i\hat{s}_I)\cdot(\hat{s}_R + i\hat{s}_I) = 1 \Rightarrow \underbrace{s_R^2}_{\hat{s}_R\cdot\hat{s}_R} - \underbrace{s_I^2}_{\hat{s}_I\cdot\hat{s}_I} = 1,\ \hat{s}_R\cdot\hat{s}_I = 0$$

This can be satisfied if, for example, we have:

$$\hat{s}_R = \cosh u\,\hat{x}, \quad \hat{s}_I = \sinh u\,\hat{y}$$

Then:

$$e^{i\vec{k}\cdot\hat{x}} = e^{\frac{i\omega}{c} n \cosh u\, x}\, e^{-\frac{\omega}{c} n \sinh u\, y}$$

This represents an <u>inhomogeneous plane wave</u> that propagates in the x direction and (for $u>0$) decays in the y direction. This behavior occurs, for example, in <u>total internal reflection</u>.

<u>Polarization</u>

Parameters of a plane wave are its frequency ω, direction of propagation $\hat{k}$, and its polarization specified by $\vec{E}_0$ (note that $\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0$). Let us choose $\vec{k}$ to be real and in the z direction.

Then:

$\vec{E} = \vec{E}_0 e^{i(kz-\omega t)}$

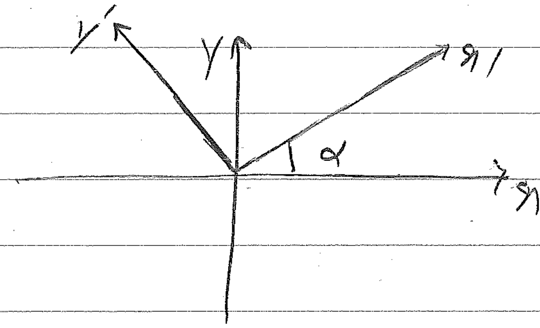
Where $\vec{E} \cdot \vec{k} = 0$ implies that:

$\vec{E}_0 = (E_x \hat{x} + E_y \hat{y})$

In general, the following cases are possible:

(1) $\frac{E_x}{E_y}$ is real. This happens if E_x and E_y have the same phase or are exactly out of phase. The wave in this case is linearly polarized.

A rotation of the x and y axes by angle α results in:



$\hat{x}' = \hat{x}\cos\alpha + \hat{y}\sin\alpha$, $\hat{y}' = -\hat{x}\sin\alpha + \hat{y}\cos\alpha$

By choosing α appropriately, $\vec{E}$ can be situated entirely along the x' or y' axis.

(2) $\frac{E_x}{E_y} = \pm i$. In this case, E_x and E_y have the same magnitude

but their phases differ by $\pm \frac{\pi}{2}$. The wave is then <u>circularly</u> polarized:

$$\vec{E} = E_0 (\hat{x} \pm i\hat{y}) e^{i(kz-\omega t)}$$

The tip of the electric field (at a fixed z) moves with angular frequency ω on a circle of radius E_0 in a counterclockwise (for the + sign) or clockwise (for the - sign) manner. One can define circular polarization basis vectors as follows:

$$\hat{e}_\pm = \frac{1}{\sqrt{2}} (\hat{x} \pm i\hat{y})$$

Where:

$$\hat{e}_+ \cdot \hat{e}_+^* = \hat{e}_- \cdot \hat{e}_-^* = 1 \quad , \quad \hat{e}_+ \cdot \hat{e}_-^* = 0$$

A rotation in the xy plane by angle α results in:

$$\hat{e}'_\pm = \frac{1}{\sqrt{2}} (\hat{x}' \pm i\hat{y}') = \cos\alpha \, \hat{e}_\pm \mp i\sin\alpha \, \hat{e}_\pm = e^{\mp i\alpha} \hat{e}_\pm$$

The new basis vectors are just simply rotated by a phase $\mp\alpha$.

(3) $\frac{E_x}{E_y} = \pm ir$ ($r \neq 1$ and real). This represents a wave that has <u>elliptical</u> polarization. The tip of the electric field in this case

rotates on an ellipse whose axes coincide with the x and y axes, and the ratio of the major to minor axes is r (if $r>1$) or $\frac{1}{r}$ (if $r<1$).

(4) $\frac{E_x}{E_y} \neq ir$, real. This is the most general case, which can be parametrized as follows:

$$\boxed{\vec{E} = E_0\left(\cos\frac{\theta}{2}\,\hat{x} + e^{i\varphi}\sin\frac{\theta}{2}\,\hat{y}\right)e^{i(kz-\omega t)}}$$

Here, ϕ is the phase difference between the x and y components of $\vec{E}$ and $\tan\frac{\theta}{2}$ is the ratio of their magnitudes. This describes <u>elliptical</u> polarization with the ellipse tilted relative to the x and y axes.

We can write different polarizations in the circular polarization basis (which is often more convenient):

$$\vec{E} = E_0\,(\alpha\,\hat{e}_+ + \alpha^*\,\hat{e}_-)\,e^{i(kz-\omega t)} \qquad \text{(Linear polarization)}$$

$$\vec{E} = E_0\,\hat{e}_\pm\,e^{i(kz-\omega t)} \qquad \text{(Circular polarization)}$$

$$\vec{E} = E_0 (a\hat{e}'_+ + b\hat{e}'_-)\, e^{i(kz-\omega t)}$$ (Elliptical polarization)

($\frac{a}{b}$ real)

The polarization ellipse (in the last case) has major and minor axes that are coincident with the x' and y' axes.

Looking at the most general case, (4) in above, it is easy to see that circular and linear polarizations are special cases of the general case:

$$\vec{E} = E_0 \left(\cos\frac{\theta}{2}\hat{x} + e^{i\phi} \sin\frac{\theta}{2}\hat{y} \right) e^{i(kz-\omega t)}$$

$\phi = \pm\frac{\pi}{2}$: elliptical polarization with the major and mior axes coinciding with the x and y axes

$\phi = \pm\frac{\pi}{2}$, $\theta = \pm\frac{\pi}{2}$: circular polarization

$\phi = 0$: linear polarization