Lecture 13 02/27/2019

Maxwell Equations and Electrodynamics (Cont'd)

Conservation Laws for Energy and Momentum

Let us start from the vector identity:

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$

Maxwell equations involving $\vec{\nabla}\times\vec{E}$ and $\vec{\nabla}\times\vec{H}$ result in:

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right) \Rightarrow \int_{V}\vec{J}\cdot\vec{E}\, d^3x =$$

$$-\int_{V}\vec{\nabla}\cdot(\vec{E}\times\vec{H})\, d^3x - \int_{V}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}+\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right) d^3x$$

In general, for both linear and non-linear media, we have:

$$\frac{d}{dt}\Big(\underbrace{U_E}_{\substack{\text{electric}\\ \text{energy inside}\\ \text{volume } V}}+\underbrace{U_M}_{\substack{\text{magnetic}\\ \text{energy inside}\\ \text{volume } V}}\Big) = -\int_{V}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}+\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right) d^3x$$

Then, defining the Poynting vector $\vec{S}\equiv\vec{E}\times\vec{H}$, we find:

$$\frac{d}{dt}(U_E+U_M) = -\int_{V}\vec{\nabla}\cdot\vec{S}\, d^3x - \int_{V}\vec{J}\cdot\vec{E}\, d^3x = -\oint_{S}\vec{S}\cdot\hat{n}\, da \; -$$

$$\int_V \vec{J}\cdot\vec{E}\, d^3x \Rightarrow \boxed{\frac{d}{dt}(U_E+U_M)=\oint_S \vec{S}\cdot(-\hat{n})\, da-\int_V \vec{J}\cdot\vec{E}\, d^3x}$$

The term on the left-hand side represents the time variation of the energy in the electromagnetic field within volume V. The first term on the right-hand side is the flux of electromagnetic energy into V, and the second term is the rate of work done by the field ($\vec{E}$ field only) on the charges. The Poynting vector $\vec{S}$ represents the energy flux per unit time per unit area normal to it. Note that the power flowing in and out of a volume V is given by the surface integral of $\vec{S}$ over the boundary of V.

The differential form of the energy conservation is:

$$\frac{\partial}{\partial t}(u_E+u_M)=-\vec{\nabla}\cdot\vec{S}-\vec{J}\cdot\vec{E}$$

Where, for linear media, we have:

$$u_E=\frac{1}{2}\vec{E}\cdot\vec{D} \quad , \quad u_M=\frac{1}{2}\vec{B}\cdot\vec{H}$$

However, these relations are not valid for non-linear media.

We can also derive an expression for momentum conservation.

Recall that for a number n of point charges:

$$\frac{d\vec{P}_{mech}}{dt} = \sum_{i=1}^{n} q_i \left(\vec{E}_i + \vec{v}_i \times \vec{B}_i\right)$$

For a general distribution of charge ρ and current $\vec{J}$, we have:

$$\frac{d\vec{P}_{mech}}{dt} = \int_V \left(\rho \vec{E} + \vec{J} \times \vec{B}\right) d^3 r$$

Assuming that there are no bound charges or currents $\rho = \varepsilon_0 (\vec{\nabla} \cdot \vec{E})$

and $\vec{J} = \frac{1}{\mu_0}\left[\vec{\nabla} \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right]$. Hence:

$$\frac{d\vec{P}_{mech}}{dt} = \int_V \left[\varepsilon_0 (\vec{\nabla} \cdot \vec{E})\vec{E} + \frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) \times \vec{B} - \varepsilon_0 \left(\frac{\partial \vec{E}}{\partial t} \times \vec{B}\right)\right] d^3 r$$

After using $\frac{\partial \vec{E}}{\partial t} \times \vec{B} = \frac{\partial}{\partial t}(\vec{E} \times \vec{B}) - \vec{E} \times \frac{\partial \vec{B}}{\partial t}$, we find:

$$\frac{d\vec{P}_{mech}}{dt} + \frac{d}{dt}\int_V \varepsilon_0 (\vec{E} \times \vec{B})\, d^3 r = \int_V \Big[\varepsilon_0 (\vec{\nabla} \cdot \vec{E})\vec{E} + \varepsilon_0 \underbrace{(\vec{\nabla} \times \vec{E})}_{= -\frac{\partial \vec{B}}{\partial t}} \times \vec{E}$$

$$+ \frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) \times \vec{B}\Big] d^3 r$$

We can make the right-hand side symmetric with respect to

$\vec{E}$ and $\vec{B}$ by adding the term $\frac{1}{\mu_0}(\vec{\nabla} \cdot \vec{B})\vec{B}$ since $\vec{\nabla} \cdot \vec{B} = 0$. Then:

$$\frac{d}{dt}\left(\vec{P}_{mech}+\vec{P}_{EM}\right)=\hat{e}_i\int_V \partial_j T_{ij}\, d^3x$$

Where P_{EM} is the momentum of the electromagnetic field within volume V defined as:

$$\vec{P}_{EM}=\int_V \varepsilon_0(\vec{E}\times\vec{B})\,d^3x \qquad \left(\varepsilon_0(\vec{E}\times\vec{B}):\text{ momentum density}\right)$$

And T_{ij} is the Maxwell stress tensor given by:

$$T_{ij}=\varepsilon_0\left[E_iE_j+c^2B_iB_j-\frac{\delta_{ij}}{2}\left(E^2+c^2B^2\right)\right]$$

Momentum Conservation can also be written as:

$$\boxed{\frac{d}{dt}\left(\vec{P}_{mech}+\vec{P}_{EM}\right)=\hat{e}_i\oint_S T_{ij}\,n_j\,da}$$

The left-hand side represents the rate at which the total momentum changes, while the right-hand side can be interpreted as the total force on the combined (i.e., fields plus charges) system.

Example: Plane wave. In this case (as we will see later), we have:

$$\vec{B}=\frac{1}{c}\hat{k}\times\vec{E}\,, \qquad \hat{k}=\frac{\vec{k}}{k}\text{ : unit vector along propagation direction}$$

Therefore:

$$\vec{P}_{EM} = \varepsilon_0 \times \frac{1}{c}\int_V \vec{E}\times(\hat{k}\times\vec{E})\,d^3r = \frac{1}{c}\int_V \varepsilon_0\left[(\vec{E}\cdot\vec{E})\hat{k} - \overset{0}{\cancel{(\vec{E}\cdot\hat{k})}}\vec{E}\right]d^3r$$

$$\Rightarrow \vec{P}_{EM} = \frac{1}{c}\int_V \varepsilon_0 E^2\,d^3r\;\hat{k}$$

Note that for a plane wave, we have:

$$B = \frac{E}{c} \Rightarrow \frac{1}{2}\frac{B^2}{\mu_0} = \frac{1}{2}\;\frac{E^2}{\mu_0 c^2} = \frac{1}{2}\varepsilon_0 E^2$$

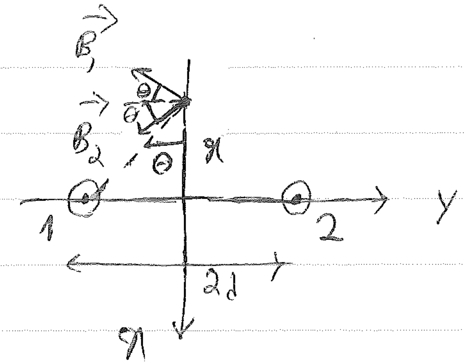
Thus:

$$u_{EM} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0} = \varepsilon_0 E^2 \Rightarrow \boxed{\vec{P}_{EM} = \frac{U_{EM}}{c}\hat{k}}$$

This is compatible with particle interpretation of a plane wave as a collection of photons for which $p = \frac{E}{c}$.

Example: Force between two parallel wires of infinite length carrying the same current I.

Let us choose coordinate axes such that the top view of the wires looks like as follows. The wires are then in the z direction.



In order to find the force on wire 2, $\vec{F}$, we choose an infinite box $y \geq 0$, $-\infty < x < +\infty$ and $0 \leq z \leq 1$ that contains unit length of that wire. Note that $\vec{P}_{EM} = 0$ since $\vec{E} = 0$. Hence:

$$F_i = \frac{dP_{mech,i}}{dt} = \oint_S T_{ij} \, n_j \, da$$

Since $\vec{E} = 0$, we have:

$$T_{ij} = \frac{1}{\mu_0}\left[B_i B_j - \frac{1}{2}\delta_{ij} B^2\right]$$

Along the x axis:

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = 2\frac{\mu_0 I}{2\pi\sqrt{d^2+x^2}} \underbrace{\cos\theta}_{=\frac{x}{\sqrt{d^2+x^2}}} (-\hat{y}) = \frac{-\mu_0 I x}{\pi(d^2+x^2)}\hat{y}$$

The only relevant surface for the integral $\oint_S$ is $0 \leq z \leq 1$ and $-\infty < x < \infty$, for which:

$$B_2 = \frac{-\mu_0 I x}{\pi(d^2+x^2)}, \quad B_1 = B_3 = 0$$

Hence:

$$T_{11} = T_{33} = -\frac{1}{2\mu_0}B^2 = -\frac{1}{2\mu_0}\left(\frac{\mu_0 I x}{\pi(d^2+x^2)}\right)^2, \quad T_{22} = \frac{1}{\mu_0}\left(B_2^2 - \frac{1}{2}B^2\right) = \frac{1}{2\mu_0}\left(\frac{\mu_0 I x}{\pi(d^2+x^2)}\right)^2$$

$T_{ij} = 0$ for $i \neq j$

This implies that F_2 is the only non-zero component of $\vec{F}$, which is given by:

$$F_2 = \oint_S T_{2j} n_j \, da = -\int_{-\infty}^{+\infty} T_{22}(x)\, dx \int_0^1 dz = \frac{-\mu_0 I^2}{2\pi^2} \int_{-\infty}^{+\infty} \frac{x^2}{(d^2+x^2)^2}\, dx$$

$$x = d\tan\theta \Rightarrow F_2 = \frac{-\mu_0 I^2}{\pi^2 d} \int_0^{\frac{\pi}{2}} \frac{\tan^2\theta \sec^2\theta}{\sec^4\theta}\, d\theta = \frac{-\mu_0 I^2}{\pi^2 d} \int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta$$

$$\Rightarrow F_2 = \frac{-\mu_0 I^2}{4\pi d} \Rightarrow \boxed{\vec{F} = \frac{-\mu_0 I^2}{4\pi d} \hat{y}}$$

This is the same as the result that we found earlier (and in a much simpler way!).