Lecture 10 02/18/2019

Review of Magnetostatics (Cont'd)

Consider situations such that $\vec{J}=0$ (i.e., no free current) but $\vec{M}\neq 0$ (i.e., magnetic materials). Then:

$\vec{\nabla}\times\vec{H}=\vec{J}=0 \Rightarrow \vec{H}=-\vec{\nabla}\Phi_M$, Φ_M: magnetic scalar potential

But:

$\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \vec{\nabla}\cdot(\mu_0\vec{H}+\mu_0\vec{M})=0 \Rightarrow \vec{\nabla}\cdot H=-\vec{\nabla}\cdot\vec{M}$

Defining $\rho_M\equiv-\vec{\nabla}\cdot\vec{M}$, results in:

$$\nabla^2\Phi_M=-\rho_M$$

ρ_M is defined as "effective magnetic charge". Note that this is just a mathematically useful relation as there is no real magnetic charge (no "magnetic monopole"). ρ_M makes sense in analogy to electrostatics since we now have a Poisson equation for Φ_M that can be solved by using the same techniques as in electrostatics.

For a general shape of the boundary of a magnetic material, there is also a surface magnetic charge density, similar to dielectrics in electrostatics, given by:

$$\vec{\sigma}_M = \vec{M} \cdot \hat{n}$$

For example, for a uniformly magnetized bar $\vec{M} = M\hat{z}$, we have $\rho_M = 0$ and $\sigma_M = \pm M$ at the two ends.

In the absence of boundaries, the sloution to the Poisson equation $\nabla^2 \Phi_M = -\rho_M$ is:

$$\Phi_M(\vec{x}) = \frac{1}{4\pi}\int \frac{\rho_M(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d\sigma' = -\frac{1}{4\pi}\int \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d\sigma' = -\frac{1}{4\pi}\int \Big[\vec{\nabla}' \cdot \left(\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) - \vec{M}(\vec{x}') \cdot \vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\Big]\, d\sigma' = -\frac{1}{4\pi}\oint_S \cancel{\frac{\overbrace{\vec{M}(\vec{x}') \cdot \hat{n}'}^{0}}{|\vec{x}-\vec{x}'|}}\, da'$$

$$+ \frac{1}{4\pi}\int \vec{M}(\vec{x}') \cdot -\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d\sigma' = -\frac{1}{4\pi}\vec{\nabla} \cdot \int \frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d\sigma'$$

For a localized distribution, at large distances we have:

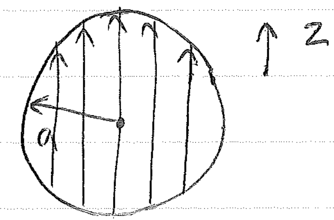
$$\frac{1}{|\vec{x}-\vec{x}'|} \simeq \frac{1}{|\vec{x}|} \Rightarrow \Phi_M(\vec{x}) \simeq -\frac{1}{4\pi}\vec{\nabla} \cdot \int \frac{\vec{M}(\vec{x}')}{|\vec{x}|}\, d\sigma' = \frac{\vec{m} \cdot \vec{x}}{4\pi r^3}$$

$$\vec{m} = \int \vec{M}(\vec{x}')\, d\sigma'$$

We note that this is consistent with having no "magnetic monopoles"; as the first non-vanishing contribution is a dipole term.

Example: Uniformly magnetized sphere.

$$\vec{M}_s \begin{cases} M\hat{z} & r \le a \\ 0 & r > a \end{cases}$$



In this case $\sigma_M = \vec{M}\cdot\hat{n} = M\cos\theta$. Thus:

$$\Phi_M(\vec{x}) = \frac{1}{4\pi}\iint_{\theta'\,\phi'} \frac{M\cos\theta'\, a^2}{|\vec{x}-\vec{x}'|}\, d\Omega'$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m} \frac{r_<^l}{r_>^{l+1}} \frac{4\pi}{2l+1} \cdot Y_{lm}(\theta,\phi)\, Y_{lm}^*(\theta',\phi')$$

$$\cos\theta' = \sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta',\phi')$$

Orthogonality of Y_{lm}'s implies that only the $l=1, m=0$ term in above has a non-zero contribution. Therefore:

$$\Phi_M(\vec{x}) = \frac{Ma^2}{3}\frac{r_<}{r_>^2}\cos\theta$$

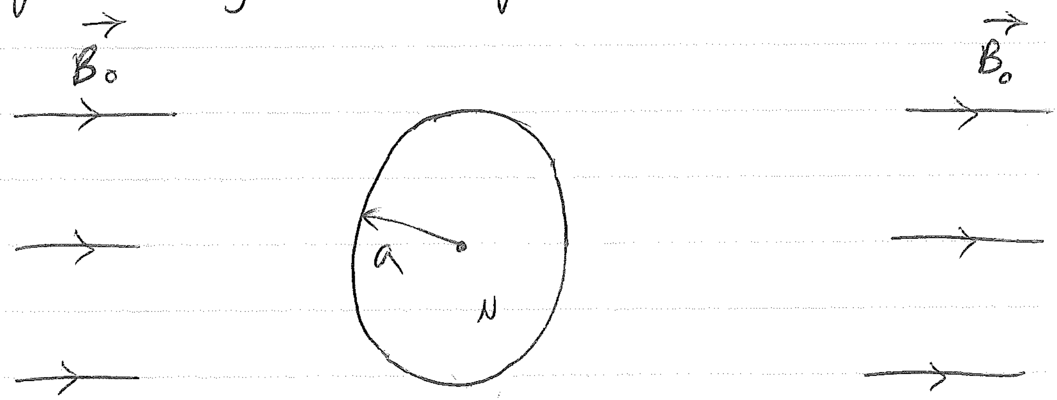
For $r<a$, we have $r_< = r$ and $r_> = a$. Hence:

$$\Phi_M(\vec{x}) = \frac{M r\cos\theta}{3} = \frac{Mz}{3} \Rightarrow \boxed{\vec{H} = -\vec{\nabla}\Phi_M = -\frac{\vec{M}}{3}}$$

For $r > a$, we have $r_< = a$ and $r_> = r$, which results in:

$$\boxed{\Phi_M(\vec{x}) = \frac{Ma^3}{3r^2}\cos\theta = \frac{m\cos\theta}{4\pi r^2}, \quad m = \frac{4\pi}{3}a^3 M}$$

Example: A magnetizable sphere in an external uniform $\vec{B}$ field.



We can use the results for a similar problem in electrostatics, namely a dielectric sphere in a uniform electric field. The equations in the two cases are the same upon making the following correspondence:

$$\vec{E} \leftrightarrow \vec{H}, \quad \frac{\vec{D}}{\varepsilon_0} \leftrightarrow \frac{\vec{B}}{\mu_0}, \quad \frac{\vec{P}}{\varepsilon_0} \leftrightarrow M$$

This results in:

$$\vec{H}_{in} = \frac{\vec{B}_0}{\mu_0} - \frac{\vec{M}}{3}$$

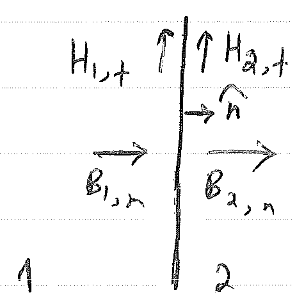
After using $\vec{B}_{in} = \mu_0 \vec{H}_{int} + \mu_0 \vec{M} = \mu \vec{H}_{in}$, we find:

$$\vec{B}_0 + \frac{2}{3}\mu_0 \vec{M} = \frac{\mu}{\mu_0}\vec{B}_0 - \frac{\mu \vec{M}}{3} \Rightarrow \vec{M}\left(\frac{2}{3}\mu_0 + \mu\right) = \left(\frac{\mu}{\mu_0} - 1\right)\vec{B}_0 \Rightarrow$$

$$\boxed{\vec{M} = \frac{3}{\mu_0}\,\frac{\frac{\mu}{\mu_0} - 1}{\frac{\mu}{\mu_0} + 2}\,\vec{B}_0}$$

Boundary Conditions at Magnetic Interfaces

Since $\vec{\nabla}\cdot\vec{B} = 0$ and $\vec{\nabla}\times\vec{H} = \vec{J}$, at an interface we have:



$$B_{2,n} = B_{1,n}$$

$$\hat{n}\times(\vec{H}_{2,t} - \vec{H}_{1,t}) = \vec{K} \quad (\Rightarrow |H_{2,t} - H_{1,t}| = |\vec{K}|)$$

Here $\vec{K}$ denotes the surface density of the free current at the interface. In the absence of free currents, we have $H_{2,t} = H_{1,t}$. In terms of the $\vec{B}$ field, we have:

$$B_{2,n} = B_{1,n} \quad , \quad \hat{n}\times\left(\frac{\vec{B}_{2,t}}{\mu_2} - \frac{\vec{B}_{1,t}}{\mu_1}\right) = \vec{K}$$

Inside a perfect conductor, the $\vec{E}$ and $\vec{B}$ fields vanish in the static limit as free charges can redistibute on the surface and also form surface currents. However, non-zero polarization and magnetization fields, $\vec{P}$ and $\vec{M}$ respectively, can still exist. In the case that $\vec{E}=0$, we have $\vec{D}=\epsilon_0\vec{P}=\epsilon\vec{E}$. Hence $\vec{E}=0$ and $\vec{P}\neq 0$ is only possible if $\epsilon\to\infty$. Similarly, if $\vec{B}=0$, we have $\vec{H}=-\vec{M}=\frac{\vec{B}}{\mu}$. Therefore $\vec{B}=0$ is consistent with $\vec{M}\neq 0$ only if $\mu\to 0$. This implies that a conductor can be considered as a material with permittivity ϵ and permeability μ in the limit that $\frac{\epsilon}{\epsilon_0}\to\infty$ and $\frac{\mu}{\mu_0}\to 0$.

We note that since $E_t=E_n=0$ and $B_t=B_n=0$ inside a conductor, at the interface with a conductor we have $B_n=0$ and $E_t=0$. However, E_n and B_t are in general non-zero due to the surface charge density and current density on the surface of the conductor.

Magnetic Energy

The energy stored in a magnetic field generated by a steady current arises due to the work done by the current source. The work to build a current from zero to its final value is done against the electromotive force generated by the changing $\vec{B}$ field that is produced by the current. According to Faraday's law, the electromotive force is given by:

$$\mathcal{E} = \oint_C \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt} \quad , \quad \underbrace{\Phi_B}_{\text{magnetic flux}} = \int_S \vec{B} \cdot \hat{n}\, da$$

The work done per unit time to increase the current from I to $I + dI$ is:

$$\frac{dW}{dt} = -I\mathcal{E} = I\frac{d\Phi_B}{dt} \Rightarrow \delta W = I\,\delta\Phi_B$$

$$\delta\Phi_B = \delta\int_S \vec{B}\cdot\hat{n}\,da = \delta\int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,da = \int_S (\vec{\nabla}\times\delta\vec{A})\cdot\hat{n}\,da =$$

$$\oint_C \delta\vec{A}\cdot d\vec{l}$$

For a general current distribution, then the last expression becomes:

$$\delta W = \int_V (\delta\vec{A}\cdot\vec{J})\,d\tau \Rightarrow \delta W = \int_V \delta\vec{A}\cdot(\vec{\nabla}\times\vec{H})\,d\tau = \int_V [-\vec{\nabla}\cdot(\delta\vec{A}\times\vec{H}) +$$

$$\vec{H}\cdot(\vec{\nabla}\times\delta\vec{A})]\,d\tau$$

$$\int_V \vec{\nabla}\cdot(\delta\vec{A}\times\vec{H})\,d\tau = \oint_S (\delta\vec{A}\times\vec{H})\cdot\hat{n}\,da = 0$$

for a localized distribution

Thus:

$$\delta W = \int \vec{H}\cdot(\vec{\nabla}\times\delta\vec{A})\,d\tau = \int \vec{H}\cdot\delta\vec{B}\,d\tau = \int \frac{1}{2}\,\delta(\vec{H}\cdot\vec{B})\,d\tau \Rightarrow$$

$$\boxed{W = \int \frac{1}{2}(\vec{H}\cdot\vec{B})\,d\tau}$$

In vacuum, the magnetic energy density is $\frac{1}{2}\frac{|\vec{B}|^2}{\mu_0}$. While, in a magnetic material we have $\frac{1}{2}\frac{|\vec{B}|^2}{\mu}$.