Physics 511: Electrodynamics

Spring 2019

Midterm Exam #2

April 10, 2019

Instructions:

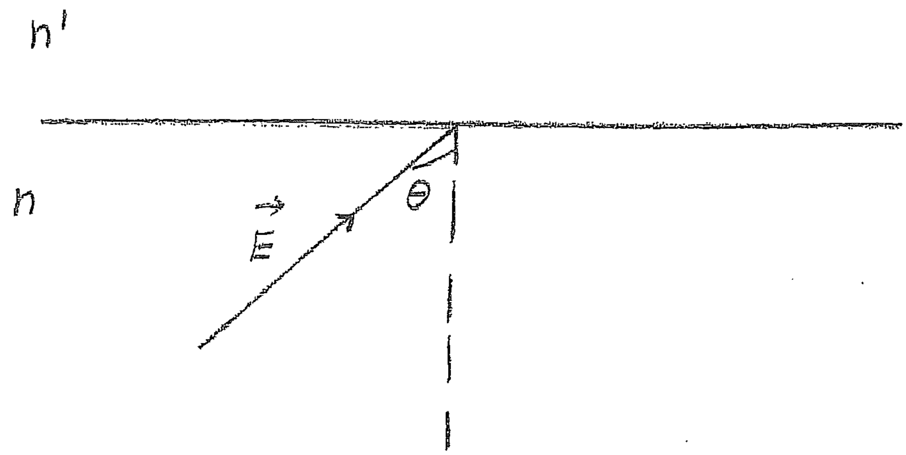
- Do any 1 of problems 1 and 2, and any 1 of problems 3 and 4. All problems carry the same weight.
- This is an open-book open-note exam.

1- A plane wave of frequency ω is incident at an angle θ at the planar boundary between two non-magnetic lossless media with refraction indices n and n', where $n < n'$, as shown in the figure. The wave has positive circular polarization $\vec{E} = E\hat{e}_+$ (using coomplex notation).

(a) Write the electric field vector of the incident wave in a basis consisitng of unit vectors $\hat{s}$ and $\hat{p}$ corresponding to s and p polarizations rescpetively.

(b) Find the electric field vector of the reflected wave and describe its polarization. How does the sense of rotation of the electric field vector (i.e., clockwise or counter clockwise) varies with the incident angle θ?

(c) Find all incident angles for which the reflected wave is linearly or circularly polarized.



2- A plane wave of frequency ω is incident at angle θ from vacumm on the front surface of a plane-parallel dielectric non-magnetic slab of thickness d and index of refraction n (where n is real). The back surface of the slab is coated with a *perfectly* conducting film. Assume that the wave has s polarization.

(a) Write down the transfer matrix for the vacuum-dielectric interface as well as the propagation matrix for propagation in the dielectric slab out to its conducting back surface. Use the boundary conditions to relate the two traveling components of the electric field at the conducting interface.

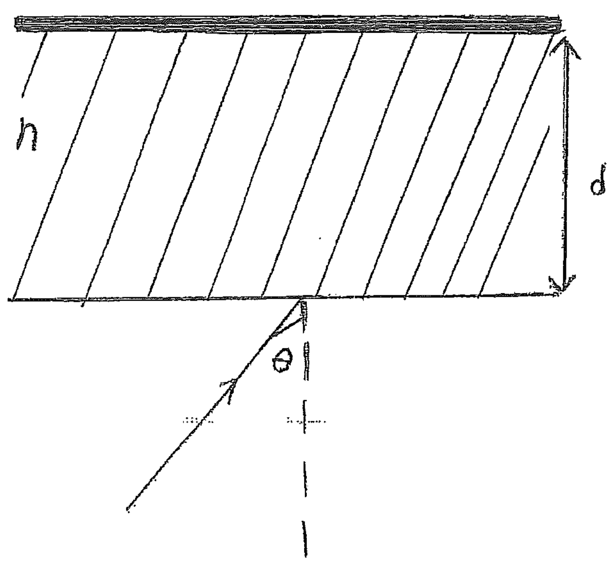
(b) Show that the amplitude reflection coefficient for the slab follows:

$$r = -\frac{r_s \exp(-i\alpha) + \exp(i\alpha)}{r_s \exp(i\alpha) + \exp(-i\alpha)},$$

where

$$r_s \equiv \frac{\sqrt{n^2 - \sin^2\theta} - \cos\theta}{\sqrt{n^2 - \sin^2\theta} + \cos\theta} \quad , \quad \alpha \equiv \frac{\omega d \sqrt{n^2 - \sin^2\theta}}{c}.$$

Show that $|r| = 1$ and interpret this result.



3- Consider a uniform plane charge of surface charge density σ in the xy plane. At time $t = 0$ the entire charge plane begins to oscillate periodically with velocity $v = v_0 \sin\omega t \; \hat{y}$.

(a) Find the current density $\vec{J}(\vec{x}, t)$. Is it fully transverse, fully longitudinal, or neither?

(b) Show that the vector potential $\vec{A}$ obeys the same equation in the Lorentz and Coulomb gauges in this case. (Hint: In the Coulomb gauge, first find ϕ.)

(c) Show that $\vec{A}$ is directed along the y axis and depends only on z and t. Use this to find the direction of $\vec{B}$ and $\vec{E}$ fields and verify that it is compatible with a wave proagating away form the plane in the z direction.

Hint: You may use the retarded Green's function for the time-dependent wave equation in one spatial dimension:

$$G(z, t; z', t') = -\frac{c}{2}\Theta(t - t' - |z - z'|/c) .$$

4- Consider the angular momentum of the electromagnetic field of a circularly polarized Bessel beam traveling in vacuum in the z direction. Such a beam is represented by the following $\vec{E}$ and $\vec{B}$ fields (in complex notation):

$$\vec{E} = \left[E_0\ \hat{e}_+ + \frac{i}{k}\left(\frac{\partial E_0}{\partial x} + i\frac{\partial E_0}{\partial y}\right)\hat{z}\right]\exp[i(kz-\omega t)] \quad , \quad \vec{B} = -\frac{i}{c}\vec{E}\,,$$

where $\hat{e}_+ = \hat{x} + i\hat{y}$ and $k \approx \omega/c$. The specific form of E_0 is best expressed in polar coordinates as

$$E_0(\rho,\phi) = A\ J_m(\gamma\rho)\ \exp(im\phi)\,,$$

where J_m is the Bessel function of the first kind of order m and $\gamma \ll k$.

(a) Show that the z component of the time-averaged total angular monetum may be expressed as follows:

$$\langle J_z\rangle = \frac{\epsilon_0}{2}\ \mathrm{Re}\int d^3x\ (\hat{z}\times\vec{x})\cdot(\vec{E}\times\vec{B}^*)\,.$$

(b) Show that the expression in part (a) may be written as

$$\langle J_z\rangle = \frac{\epsilon_0}{2ck}\int\left(m|E_0|^2 - \frac{\rho}{2}\frac{\partial|E_0|^2}{\partial\rho}\right)d^3x\,.$$

You may use the following result without proof:

$$\left(\frac{\partial E_0}{\partial x} + i\frac{\partial E_0}{\partial y}\right) = e^{i\phi}\left(\frac{\partial E_0}{\partial\rho} + \frac{i}{\rho}\frac{\partial E_0}{\partial\phi}\right)\,.$$

(c) By performing an integration by parts over ρ, show that the result of part (b) may be expressed as

$$\langle J_z\rangle = \frac{\epsilon_0}{2ck}(m+1)\int|E_0|^2 d^3x\,.$$

Interpret this result in terms of photon angular momentum.