PHYC 467: Methods of Theoretical Physics II

Spring 2013

Homework Assignment #5

(Due March 28, 2013)

1- The adjoint representation of $SU(3)$ is given by 8 generators U_i $(1 \leq i \leq 8)$ where $(U_i)_{jk} = if_{ijk}$. Show that the following relations hold for the two Casimir operators $C_1 = \sum_i U_i^2$ and $C_2 = \sum_{ijk} d_{ijk} U_i U_j U_k$:

$$C_1 = 3I \quad , \quad C_2 = 0 \, ,$$

where I is the identity operator and $d_{ijk} = \mathrm{Tr}([U_i, U_j]_+ U_k)/4$.

Hint: The non-vanishing f_{ijk} are:

$$f_{123} = 1 \ , \ f_{147} = \frac{1}{2} \ , \ f_{156} = -\frac{1}{2} \ , \ f_{246} = \frac{1}{2} \ , \ f_{257} = \frac{1}{2} \, ,$$
$$f_{345} = \frac{1}{2} \ , \ f_{367} = -\frac{1}{2} \ , \ f_{458} = \frac{\sqrt{3}}{2} \ , \ f_{678} = \frac{\sqrt{3}}{2} \, ,$$

and those obtained by permutation of the indices. You may also use the following relations:

$$\sum_{ik} f_{ijk} f_{ikl} = -3\delta_{jl} \, ,$$
$$\sum_{m} f_{pkm} d_{mlq} = -\sum_{m} f_{qkm} d_{mlp} - \sum_{m} f_{lkm} d_{mpq} \, .$$

2- Show that the Casimir operator $C_1 = \sum_i F_i^2$ in the representation $D(p,q)$ is given by:

$$C_1 = \left(\frac{p^2 + pq + q^2}{3} + p + q \right) I \, . \qquad (1)$$

Verify that for the adjoint representation this gives you the same result as that in problem 1.

Hint: You may write C_1 in terms of T_+, T_-, V_+, V_-, U_+, U_-, T_3, Y and then use the state with maximal weight to evaluate it.