PHYC 467: Methods of Theoretical Physics II

Spring 2013

Homework Assignment #4

(Due March 5, 2013)

1- One can show that two of the Casimir operators of a semi-simple Lie algebra are given by

$$C_1 = \sum_{ijk} f_{ijk} L_i L_j L_k \,,$$
$$C_2 = \sum_{ijk} d_{ijk} L_i L_j L_k \,,$$

where

$$f_{ijk} = -iC_{ijk} \quad , \quad d_{ijk} = \mathrm{Tr}([L_i, L_j]_+ L_k)\,.$$

Here C_{ijk} are structure constants of the algebra and $[L_i, L_j]_+ \equiv L_i L_j + L_j L_i$ denotes the anti-commutator of generators.

(a) Show that d_{ijk}'s are symmetric in all indices.

(b) Assuming that generators are normalized, show that $C_1 \propto \sum_i L_i^2$.

(c) Consider the $SU(2)$ algebra. Explicitly verify that C_1 is a Casimir operator.

(d) Show that $C_2 = 0$ in the case of $SU(2)$ algebra, which is in agreement with Racah's theorem.

2- Consider the adjoint representation of $SU(2)$ in which $(J_i)_{jk} = C_{ijk}$.

(a) Explicitly write down the $SU(2)$ generators in this representation.

(b) Find a basis in which J^2 and J_3 are diagonal. What do J_1, J_2 look like in this basis?

3- Consider 3-dimensional representation of the $SU(2)$ algebra (corresponding to a spin-1 particle). Show that the direct product of two such representations can be decomposed into 5-dimensional, 3-dimensional, and 1-dimensional representations. Find the Clebsch-Gordan coefficients for the 5-dimensional representation corresponding to a spin-2 particle.