PHYC 467: Methods of Theoretical Physics II

Spring 2013

Homework Assignment #3

(Due February 26, 2013)

1- There is an elementary criterion for finding out whether a Lie algebra is semisimple which originates from Cartan. One can define the symmetric tensor, called a metric tensor, built from structure constants of the algebra:

$$g_{\sigma\lambda} = C_{\sigma\rho\tau} C_{\lambda\tau\rho} \, .$$

Show that a Lie algebra is semisimple if $\det(g_{\sigma\lambda}) \neq 0$. Apply Cartan's criterion to the $SU(2)$ and $U(2)$ algebras and verify their semisimplicity.
(Hint: Choose the basis in which the generators are normalized. Show that if the algebra is not semisimple, then $\det(g_{\sigma\lambda}) = 0$.)

2- The translation-rotation group in three dimensions (the Euclidean group E_3) has 6 generators obeying the following commutation relations:

$$[J_i, J_j] = \epsilon_{ijk} J_k \quad , \quad [P_i, P_j] = 0 \quad , \quad [P_i, J_j] = \epsilon_{ijk} P_k \, .$$

Using these commutation relations, show that the two operators $|\vec{P}|^2$, $\vec{J} \cdot \vec{P}$ are Casimir operators of E_3.

3- The adjoint representation (or regular representation) of an algebra is defined as the representation in which the matrix elements of generators are given by $(L_i)_{jk} = C_{ijk}$. Assuming normalized generators, show that $\det(L_i) = \det(-L_i)$ in the adjoint representation. This implies that L_i and $-L_i$ may be related to each other via a similarity transformation in the adjoint reprentation.