PHYC 467: Methods of Theoretical Physics II

Spring 2013

Homework Assignment #2

(Due February 14, 2013)

1- Let A and B be two complex $n \times n$ matrices. Show that:

(a) If $[A, B] = 0$, then

$$e^A \; e^B = e^{A+B} \, .$$

(b) If B is invertible, then

$$B \; e^A \; B^{-1} = e^{BAB^{-1}} \, .$$

(c) If $\lambda_1, \; ..., \; \lambda_n$ are the eigenvalues of A, then $e^{\lambda_1}, \; ..., \; e^{\lambda_n}$ are the eigenvalues of $\exp(A)$.

(d)

$$[e^A]^* = e^{A^*} \quad , \quad [e^A]^T = e^{A^T} \, ,$$
$$[e^A]^\dagger = e^{A^\dagger} \quad , \quad [e^A]^{-1} = e^{-A} \, .$$

(e)

$$\det e^A = e^{\mathrm{Tr} A} \, .$$

2- Generators L_i of a Lie group can be normalized such that $\mathrm{Tr}(L_i L_j) = \delta_{ij}/2$. Show that the resulting structure constants are purely imaginary and totally antisymmetric in this case. Verify this in the matrices $\sigma_1/2, \; \sigma_2/2, \; \sigma_3/2$ are normalized generators of the $SU(2)$ group, where $\sigma_1, \; \sigma_2, \; \sigma_3$ denote the Pauli matrices.

3- Let $a_1^\dagger,\ a_1$ and $a_2^\dagger,\ a_2$ denote two sets of creation and annihilation operators that satisfy the following commutation relations:

$$[a_i, a_j^\dagger] = \delta_{ij} \quad , \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0\,.$$

(a) Show that the three operators

$$T_1 = a_1^\dagger a_2 + a_2^\dagger a_1 \quad , \quad T_2 = -i(a_1^\dagger a_2 - a_2^\dagger a_1) \quad , \quad T_3 = a_1^\dagger a_1 - a_2^\dagger a_2\,,$$

satisfy the same commutation relations as the Pauli matrices, and hence generate the $SU(2)$ algebra.

(b) Show that the number operator defined as $N = a_1^\dagger a_1 + a_2^\dagger a_2$ is a Casimir operator of this $SU(2)$ algebra.