PHYC 467: Methods of Theoretical Physics II

Spring 2013

Midterm Exam

- This is a take home exam. Distributed on Tuesday March 5, due on Thursday March 7 by 4:00 pm.
- All reference material allowed, but no DISCUSSING PROBLEMS.
- Any questions are to be directed to the instructor.
- Two problems, equally weighted.

1- Consider the twelve operators $L_{\alpha\beta}$ ($1 \leq \alpha, \beta \leq 4$ and $L_{\beta\alpha} = -L_{\alpha\beta}$) that obey the following commutation relations:

$$[L_{\alpha\beta}, L_{\mu\nu}] = i(\delta_{\alpha\mu}L_{\beta\nu} + \delta_{\beta\nu}L_{\alpha\mu} - \delta_{\alpha\nu}L_{\beta\mu} - \delta_{\beta\mu}L_{\alpha\nu}) \, .$$

The operators $L_{12}, \ L_{23}, \ L_{31}, \ L_{14}, \ L_{24}, \ L_{34}$ together with their commutation relations constitute the $SO(4)$ algebra. For simplicity, we define $L_1 = L_{23}, \ L_2 = L_{31}, \ L_3 = L_{12}, \ M_1 = L_{14}, \ M_2 = L_{24}, \ M_3 = L_{34}$.

(a) Show the following commutation relations ($1 \leq i, j, k \leq 3$) are satisfied:

$$[L_i, L_j] = i\epsilon_{ijk}L_k \quad , \quad [M_i, M_j] = i\epsilon_{ijk}L_k \quad , \quad [M_i, L_j] = i\epsilon_{ijk}M_k \, .$$

(b) Define new generators $J_i = (L_i + M_i)/2$ and $K_i = (L_i - M_i)/2$. Show that these generators obey the following commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad , \quad [K_i, K_j] = i\epsilon_{ijk}K_k \quad , \quad [J_i, K_j] = 0 \, .$$

(c) Use the result in part (b) and show that the $SO(4)$ algebra is equivalent to $SU(2) \bigotimes SU(2)$. Discuss whether $SO(4)$ is simple or semisimple, its rank, and the number of its Casimir operators.

(d) Use the equivalence between $SO(4)$ and $SU(2) \bigotimes SU(2)$ and write independent Casimir operators for $SO(4)$.

(e) Show explicitly that $C_1 = L_1^2 + L_2^2 + L_3^2 + M_1^2 + M_2^2 + M_3^2$ and $C_2 = L_1M_1 + L_2M_2 + L_3M_3$ commute with all the $SO(4)$ generators, and hence are Casimir operators. What is the relation between $C_1, \ C_2$ and the Casimir operators in part (d)?

2- Consider three irreducible representations of $SU(2)$ corresponding to $j_1 = 1$, $j_2 = 1/2$, and $j_3 = 1/2$ respectively. The direct product of these representation may describe the Hilbert space of a physical system consisting of two identical spin-1/2 fermions in a P-wave state (at the center-of-mass frame).

(a) Find the dimensionality of each representation and that of their direct product.

(b) Write the decomposition of the direct product representation as a direct sum of irreducible representations and determine their corresponding dimensionality.

(c) Explicitly construct the irreducible representation in part (b) that has the largest value of j by finding all of its Clebsch-Gordan coefficients.