PHYC 467: Methods of Theoretical Physics II

Spring 2013

Final Exam

Date and Time: 05/07/2013, 09:00-12:00

Instructions:

- This is an open-book, open-note exam. All reference material allowed.
- The exam consists of two problems, which are equally weighted.

1- The Hamiltonian of a three-dimensional isotropic harmonic oscillator is

$$H = \hbar\omega(a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 + \frac{3}{2}) . \tag{1}$$

where

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \quad , \quad [a_i, a_j^\dagger] = \delta_{ij} . \tag{2}$$

Because of the rotational invariance of the isotropic oscillator, the symmetry group for this system is $SO(3)$. The energy eigenvalues E_n are given by

$$E_n = (n + \frac{3}{2})\hbar\omega \qquad n = l + 2k \ (k \text{ an integer}) , \tag{3}$$

where $\hbar^2 l(l+1)$ is the eigenvalue of the orbital angular momentum operator $\hat{L}^2$. This implies that eigenstates with energy E_n belong to a reducible representation of $SO(3)$ rather than an irreducible representation. It might look contradictory as $SO(3)$ is a simple Lie group of rank 1 with $\hat{L}^2$ as its only Casimir operator.

The resolution lies in that the dynamical symmetry group of the isotropic oscillator in three dimensions is $SU(3)$. This comes from that fact the 8 operators

$$\begin{aligned}
&\lambda_1 = a_1^\dagger a_2 + a_2^\dagger a_1 \quad , \quad \lambda_2 = -i(a_1^\dagger a_2 - a_2^\dagger a_1) , \\
&\lambda_3 = a_1^\dagger a_1 - a_2^\dagger a_2 \quad , \quad \lambda_4 = a_1^\dagger a_3 + a_3^\dagger a_1 , \\
&\lambda_5 = -i(a_1^\dagger a_3 - a_3^\dagger a_1) \quad , \quad \lambda_6 = a_2^\dagger a_3 + a_3^\dagger a_2 , \\
&\lambda_7 = -i(a_2^\dagger a_3 - a_3^\dagger a_2) \quad , \quad \lambda_8 = \frac{1}{\sqrt{3}}(a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3) ,
\end{aligned} \tag{4}$$

obey the $SU(3)$ algebra and $[H, \lambda_i] = 0$.

(a) From the relation in Eq. (3) find the dimensionality of the reducible representation of $SO(3)$ that contains eigenstates with energy E_n.

(b) Show that the states $|n_1, n_2, n_3\rangle$ (where $0 \leq n_{1,2,3} \leq n, \ n_1 + n_2 + n_3 = n$) for which

$$\begin{aligned}
a_1|n_1, n_2, n_3\rangle = n_1^{1/2}|n_1 - 1, n_2, n_3\rangle \quad , \quad a_1^\dagger|n_1, n_2, n_3\rangle = (n_1 + 1)^{1/2}|n_1 + 1, n_2, n_3\rangle , \\
a_2|n_1, n_2, n_3\rangle = n_2^{1/2}|n_1, n_2 - 1, n_3\rangle \quad , \quad a_2^\dagger|n_1, n_2, n_3\rangle = (n_2 + 1)^{1/2}|n_1, n_2 + 1, n_3\rangle , \\
a_3|n_1, n_2, n_3\rangle = n_3^{1/2}|n_1, n_2, n_3 - 1\rangle \quad , \quad a_3^\dagger|n_1, n_2, n_3\rangle = (n_3 + 1)^{1/2}|n_1, n_2, n_3 + 1\rangle ,
\end{aligned} \tag{5}$$

are energy eigenstates of the isotropic oscillator.

(c) Show that for a given n all of the eigenstates can be found by the action of the following shift operators

$$T_\pm = \lambda_1 \pm i\lambda_2 \quad , \quad V_\pm = \lambda_4 \pm i\lambda_5 \quad , \quad U_\pm = \lambda_6 \pm i\lambda_7 \tag{6}$$

on the state $|n, 0, 0\rangle$. Then argue that these states form a $(n, 0)$ representation of $SU(3)$.

(d) Show that dimension of the $SU(3)$ irreducible representation in part (c) matches that of the $SO(3)$ reducible representation in part (a).

2- In $SU(5)$ grand unified theory (GUT) the elementary fermions (i.e., quarks, leptons and their antiparticles) belong to a $\bar{\mathbf{5}} \oplus \mathbf{10}$ representation of $SU(5)$. Fermions acquire their mass through their coupling to Higgs fields, which in the case of $SU(5)$ GUT belong to the $\mathbf{5}$ and $\bar{\mathbf{5}}$ representations.

Invariance of interactions under the $SU(5)$ transformations require that the direct product of two fermion representations and a Higgs representation contain a $\mathbf{1}$ of $SU(5)$.

(a) Form $\bar{\mathbf{5}} \otimes \mathbf{10}$ and $\mathbf{10} \otimes \mathbf{10}$ by using the Young diagrams and find their decomposition into irreducible representations.

(b) Use the results in part (a) and show that $\bar{\mathbf{5}} \otimes \mathbf{10} \otimes \bar{\mathbf{5}}$ and $\mathbf{10} \otimes \mathbf{10} \otimes \mathbf{5}$ contain a $\mathbf{1}$, while $\bar{\mathbf{5}} \otimes \mathbf{10} \otimes \mathbf{5}$ and $\mathbf{10} \otimes \mathbf{10} \otimes \bar{\mathbf{5}}$ do not. This tells you how the Higgs fields must be coupled to the fermions in $SU(5)$ GUT.

(c) Derive the decompositions $\mathbf{5}_{SU(5)} = \mathbf{2}_{SU(2)} \oplus \mathbf{1}_{SU(2)} \oplus \mathbf{1}_{SU(2)} \oplus \mathbf{1}_{SU(2)}$, $\mathbf{5}_{SU(5)} = \mathbf{3}_{SU(3)} + \mathbf{1}_{SU(3)} \oplus \mathbf{1}_{SU(3)}$ by using the Young diagrams.

(The $\mathbf{2}_{SU(2)}$ is the standard model Higgs, which behaves as $\mathbf{1} \oplus \mathbf{1}$ under the $SU(3)_C$ describing the strong interactions. The $\mathbf{3}_{SU(3)}$, which behaves as $\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$ under the $SU(2)_W$ of the electroweak interactions, becomes very heavy after GUT symmetry breaking and hence inaccessible to experiments.)